

COX RINGS AND COMBINATORICS II

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Dedicated to Ernest Borisovich Vinberg on the occasion of his 70th birthday

ABSTRACT. We study varieties with a finitely generated Cox ring. In a first part, we generalize a combinatorial approach developed in earlier work for varieties with a torsion free divisor class group to the case of torsion. Then we turn to modifications, e.g., blow ups, and the question how the Cox ring changes under such maps. We answer this question for a certain class of modifications induced from modifications of ambient toric varieties. Moreover, we show that every variety with finitely generated Cox ring can be explicitly constructed in a finite series of toric ambient modifications from a combinatorially minimal one.

1. INTRODUCTION

In [8], D. Cox associated to any non-degenerate, e.g. complete, toric variety Z a multigraded homogeneous coordinate ring $\mathcal{R}(Z)$. His construction, meanwhile a standard tool in toric geometry, allows generalization to certain non-toric varieties: for any normal variety X having only constant invertible global functions and a finitely generated divisor class group $\mathrm{Cl}(X)$, one can define a $\mathrm{Cl}(X)$ -graded Cox ring as

$$\mathcal{R}(X) := \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)),$$

see Section 2 for more details. We are interested in the case of a finitely generated Cox ring. Then one can define the total coordinate space $\overline{X} := \mathrm{Spec} \mathcal{R}(X)$; it comes with an action of the diagonalizable group $H := \mathrm{Spec} \mathbb{K}[\mathrm{Cl}(X)]$. Moreover, X turns out to be the good quotient of an open H -invariant subset $\widehat{X} \subseteq \overline{X}$ by the action of H . The quotient map $p: \widehat{X} \rightarrow X$ generalizes the concept of a universal torsor. Cox rings and universal torsors occur in different settings, for some recent work, see [7] and [9].

In [5], a combinatorial approach to varieties with torsion free divisor class group and finitely generated Cox ring was presented. The main observation is that in many cases, e.g. for projective X , the variety X is characterized by its total coordinate space \overline{X} and combinatorial data living in the grading group $\mathrm{Cl}(X)$. This leads to the concept of a “bunched ring” as defining data for X . The task then is, as in toric geometry, to read off geometric properties of the variety X from its defining combinatorial data. A first part of the present article generalizes the approach of [5] to the case of divisor class groups with torsion and obtains similar descriptions of the Picard group, the effective, moving, semiample and ample cones, singularities and the canonical divisor.

In a second part, we consider proper modifications $X_1 \rightarrow X_0$, e.g. blow ups, of varieties X_0 with finitely generated total coordinate ring. In general, it is a delicate task to describe the Cox ring of X_1 explicitly in terms of that of X_0 ; note that

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even finite generation may be lost. Our aim is to provide some systematic insight by investigating modifications $X_1 \rightarrow X_0$ induced from stellar subdivisions of toric ambient varieties. We figure out a large class of such modifications preserving finite generation of the Cox ring and show how to compute the Cox ring of X_1 from that of X_0 in these cases.

Let us make our approach a little more precise. A basic observation is that many varieties X_0 with a finitely generated Cox ring $\mathcal{R}(X_0)$, e.g. all projective ones, admit a *neat* embedding into a toric variety Z_0 such that there is a pullback isomorphism $\text{Cl}(Z_0) \rightarrow \text{Cl}(X_0)$. Then the toric universal torsor $\widehat{Z}_0 \rightarrow Z_0$ admits a simple description in terms of fans. Moreover, the inverse image \widehat{X}_0 of X_0 under $\widehat{Z}_0 \rightarrow Z_0$ is a reasonable candidate for a universal torsor over X_0 , and, similarly, the closure \overline{X}_0 of \widehat{X}_0 in the affine closure \overline{Z}_0 of \widehat{Z}_0 is a candidate for the total coordinate space of X_0 , see Corollary 2.7 for the accurate statements.

If $Z_1 \rightarrow Z_0$ is the toric modification arising from a stellar subdivision of the defining fan of Z_0 , then one may define a strict transform $X_1 \subseteq Z_1$ and ask for its universal torsor and its Cox ring. The idea is to consider, as before, the toric universal torsor $\widehat{Z}_1 \rightarrow Z_1$, the inverse image $\widehat{X}_1 \subseteq \widehat{Z}_1$ of X_1 and the closure $\overline{X}_1 \subseteq \overline{Z}_1$. These data are linked to the corresponding data for X_0 via a commutative diagram

$$\begin{array}{ccc}
 & \overline{Y}_1 & \\
 \swarrow /C & & \searrow //\mathbb{K}^* \\
 \overline{X}_1 & & \overline{X}_0 \\
 \uparrow & & \uparrow \\
 \widehat{X}_1 & & \widehat{X}_0 \\
 \downarrow /H_1 & & \downarrow /H_0 \\
 X_1 & \xrightarrow{\quad} & X_0
 \end{array}$$

where $\overline{Y}_1 \subseteq \overline{Z}_1$ is a variety coming with actions of a finite cyclic group C and the multiplicative group \mathbb{K}^* ; these data can be explicitly computed in terms of the stellar subdivision and \overline{X}_0 , see Lemma 5.3. Now a couple of technical difficulties show up. Firstly, one has to ensure that X_1 is again neatly embedded; this is done by restricting to *neat* ambient modifications in the sense of 5.4. Next, to ensure finite generation of the Cox ring $\mathcal{R}(X_1)$, we require the neat ambient modification $Z_1 \rightarrow Z_0$ to be *controlled* in the sense that H_1 permutes transitively the components of the minus cell $\overline{Y}_1^- \subseteq \overline{Y}_1$. In Theorem 5.9, we then obtain the following.

Theorem. *Let Z_1, Z_0 be \mathbb{Q} -factorial toric varieties and $\pi: Z_1 \rightarrow Z_0$ a neat controlled ambient modification for $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$. Suppose that $X_0 \subseteq Z_0$ is neatly embedded and $\overline{X}_0 \setminus \widehat{X}_0$ is of codimension at least two in \overline{X}_0 .*

- (i) *The complement $\overline{X}_1 \setminus \widehat{X}_1$ is of codimension at least two in \overline{X}_1 . If \widehat{X}_1 is normal, then $\widehat{X}_1 \rightarrow X_1$ is a universal torsor of X_1 .*
- (ii) *The total coordinate space of X_1 is the normalization of \overline{X}_1 together with the induced action of the torus H_1 .*

A similar statement holds for contractions, see Theorem 5.12. The above result shows in particular, that the Cox ring stays finitely generated under neat controlled ambient modifications. Moreover, starting with an embedding such that \overline{X}_0 is the total coordinate space of X_0 , see Corollary 3.15, it gives a way to obtain defining equations for the total coordinate space \overline{X}_1 of X_1 from those of \overline{X}_0 and the data of

the stellar subdivision. In particular, for Cox rings with a single defining relation, Proposition 7.2 gives a very explicit statement. As Cox rings without torsion in the grading group are always factorial, this leads as well to a new construction of multigraded UFDs out of given ones, compare e.g. [11] for other constructions.

As an application of the technique of ambient modifications, we show how to reduce \mathbb{Q} -factorial projective varieties X with finitely generated Cox ring, to “combinatorially minimal” ones, i.e., varieties X that do not admit a class $[D] \in \text{Cl}(X)$ such that $[D]$ generates an extremal ray of the effective cone of X and, for some representative D , all vector spaces $\Gamma(X, \mathcal{O}(nD))$, where $n > 0$ are of dimension one. It turns out that a variety is combinatorially minimal if and only if its moving cone is the whole effective cone. Our result is the following, see Theorem 6.2.

Theorem. *Every \mathbb{Q} -factorial projective variety X with finitely generated Cox ring arises from a combinatorially minimal one X_0 via a finite sequence*

$$X = X'_n - - \triangleright X_n \longrightarrow X'_{n-1} - - \triangleright X_{n-1} \longrightarrow \cdots \longrightarrow X'_0 = X_0$$

where $X'_i - - \triangleright X_i$ is a small birational transformation and $X_i \longrightarrow X'_{i-1}$ comes from a neat controlled ambient modification of \mathbb{Q} -factorial projective toric varieties.

The effect of each reduction step on the Cox ring can be explicitly calculated. Moreover, this theorem shows that the class of modifications arising from neat controlled ambient modifications is reasonably large. As an example for contraction, we treat a singular del Pezzo surface, and show that it arises via neat controlled ambient modification from the projective plane, see Example 6.10. Going in the other direction, we show that the singularity of this surface can be resolved by means of neat ambient modifications and thereby obtain the Cox ring of its minimal resolution, see Example 7.5.

CONTENTS

1. Introduction	1
2. The Cox ring of an embedded variety	3
3. Bunched rings and their varieties	10
4. Basic geometric properties	18
5. Ambient modification	24
6. Combinatorial contraction	32
7. Example: Cox rings with a single relation	37
References	41

2. THE COX RING OF AN EMBEDDED VARIETY

Here, we introduce the Cox ring of a normal variety X with finitely generated divisor class group and show how to compute the Cox ring if X is neatly embedded into a toric variety. In contrast to [5], we define the Cox ring also for the case of torsion in the divisor class group. For this, we closely follow the lines of [3, Section 3], where an analogous construction in terms of line bundles instead of divisors is performed.

In the whole paper, we work in the category of (reduced) algebraic varieties over an algebraically closed field \mathbb{K} of characteristic zero. By a point, we always mean a closed point.

Consider a normal variety X with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group $\text{Cl}(X)$. Let $\mathfrak{D} \subseteq \text{WDiv}(X)$ be a finitely generated subgroup of the

group of Weil divisors mapping onto $\text{Cl}(X)$ and consider the sheaf of \mathfrak{D} -graded algebras

$$\mathcal{S} := \bigoplus_{D \in \mathfrak{D}} \mathcal{S}_D, \quad \mathcal{S}_D := \mathcal{O}_X(D),$$

where multiplication is defined via multiplying homogeneous sections as rational functions on X . Let $\mathfrak{D}^0 \subseteq \mathfrak{D}$ be the kernel of $\mathfrak{D} \rightarrow \text{Cl}(X)$. We fix a *shifting family*, i.e., a family of \mathcal{O}_X -module isomorphisms $\varrho_{D^0}: \mathcal{S} \rightarrow \mathcal{S}$, where $D^0 \in \mathfrak{D}^0$, such that

- $\varrho_{D^0}(\mathcal{S}_D) = \mathcal{S}_{D+D^0}$ for all $D \in \mathfrak{D}$, $D^0 \in \mathfrak{D}^0$,
- $\varrho_{D_1^0+D_2^0} = \varrho_{D_2^0} \circ \varrho_{D_1^0}$ for all $D_1^0, D_2^0 \in \mathfrak{D}^0$,
- $\varrho_{D^0}(fg) = f\varrho_{D^0}(g)$ for all $D^0 \in \mathfrak{D}^0$ and any two homogeneous f, g .

To obtain such a family, take a basis D_1, \dots, D_r of \mathfrak{D} such that \mathfrak{D}^0 is spanned by $D_i^0 := a_i D_i$, where $1 \leq i \leq s$ with $s \leq r$, choose isomorphisms $\varrho_{D_i^0}: \mathcal{S}_0 \rightarrow \mathcal{S}_{D_i^0}$, and, for $D^0 = b_1 D_1^0 + \dots + b_s D_s^0$, define ϱ_{D^0} on a homogeneous f as

$$\varrho_{D^0}(f) := \varrho_{D_1^0}(1)^{b_1} \dots \varrho_{D_s^0}(1)^{b_s} f.$$

The shifting family (ϱ_{D^0}) defines a quasicoherent sheaf of ideals of \mathcal{S} , namely the sheaf \mathcal{I} generated by all sections of the form $f - \varrho_{D^0}(f)$, where f is homogeneous and D^0 runs through \mathfrak{D}^0 . Note that \mathcal{I} is a homogeneous ideal with respect to the coarsified grading

$$\mathcal{S} = \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{S}_{[D]}, \quad \mathcal{S}_{[D]} = \bigoplus_{D' \in D + \mathfrak{D}^0} \mathcal{O}_X(D').$$

Moreover, it turns out that \mathcal{I} is a sheaf of radical ideals. Dividing the $\text{Cl}(X)$ -graded \mathcal{S} by the homogeneous ideal \mathcal{I} , we obtain a quasicoherent sheaf of $\text{Cl}(X)$ -graded \mathcal{O}_X -algebras, the *Cox sheaf*: set $\mathcal{R} := \mathcal{S}/\mathcal{I}$ let $\pi: \mathcal{S} \rightarrow \mathcal{R}$ be the projection and define the grading by

$$\mathcal{R} = \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{R}_{[D]}, \quad \mathcal{R}_{[D]} := \bigoplus_{[D'] \in \text{Cl}(X)} \pi(\mathcal{S}_{[D']}).$$

Remark 2.1. Let $s: \text{Cl}(X) \rightarrow \mathfrak{D}$ be any set-theoretical section of the canonical map $\mathfrak{D} \rightarrow \text{Cl}(X)$. Then the projection $\mathcal{S} \rightarrow \mathcal{R}$ restricts to a canonical isomorphism of sheaves of $\text{Cl}(X)$ -graded vector spaces:

$$\bigoplus_{[D] \in \text{Cl}(X)} \mathcal{S}_{s([D])} \cong \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{R}_{[D]}.$$

One can show that, up to isomorphism, the graded \mathcal{O}_X -algebra \mathcal{R} does not depend on the choices of \mathfrak{D} and the shifting family, compare [3, Lemma 3.7]. We define the *Cox ring* $\mathcal{R}(X)$ of X , also called the *total coordinate ring* of X , to be the $\text{Cl}(X)$ -graded algebra of global sections of the Cox sheaf:

$$\mathcal{R}(X) := \Gamma(X, \mathcal{R}) \cong \Gamma(X, \mathcal{S})/\Gamma(X, \mathcal{I}).$$

The sheaf \mathcal{R} defines a universal torsor $p: \widehat{X} \rightarrow X$ in the following sense. Suppose that \mathcal{R} is locally of finite type; this holds for example, if X is locally factorial or if $\mathcal{R}(X)$ is finitely generated. Then we may consider the relative spectrum

$$\widehat{X} := \text{Spec}_X(\mathcal{R}).$$

The $\text{Cl}(X)$ -grading of the sheaf \mathcal{R} defines an action of the diagonalizable group $H := \text{Spec}(\mathbb{K}[\text{Cl}(X)])$ on \widehat{X} , and the canonical morphism $p: \widehat{X} \rightarrow X$ is a good quotient, i.e., it is an H -invariant affine morphism satisfying

$$\mathcal{O}_X = (p_* \mathcal{O}_{\widehat{X}})^H.$$

In the sequel, we mean by a *universal torsor* for X more generally any good quotient $q: \mathcal{X} \rightarrow X$ for an action of H on a variety \mathcal{X} such that there is an equivariant isomorphism $\iota: \mathcal{X} \rightarrow \widehat{X}$ with $q = p \circ \iota$. If the Cox ring $\mathcal{R}(X)$ is finitely generated, then we define the *total coordinate space* of X to be any affine H -variety equivariantly isomorphic to $\overline{X} = \text{Spec}(\mathcal{R}(X))$ endowed with the H -action defined by the $\text{Cl}(X)$ -grading of $\mathcal{R}(X)$.

We discuss basic properties of these constructions. As usual, we say that a Weil divisor $\sum a_D D$, where D runs through the irreducible hypersurfaces, on an H -variety is H -invariant if $a_D = a_{hD}$ holds for all $h \in H$. Moreover, we say that a closed subset $B \subseteq Y$ of a variety Y is small, in Y if it is of codimension at least two in Y .

Proposition 2.2. *Let X be a normal variety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group. Let \mathcal{R} be a sheaf of $\text{Cl}(X)$ -graded algebras as constructed before, assume that the Cox ring $\mathcal{R}(X)$ is finitely generated, denote by $p: \widehat{X} \rightarrow X$ the associated universal torsor and by $X' \subseteq X$ the set of smooth points.*

- (i) *The Cox ring $\mathcal{R}(X)$ is normal and every homogeneous invertible $f \in \mathcal{R}(X)$ is constant.*
- (ii) *The canonical morphism $\widehat{X} \rightarrow \overline{X}$ is an open embedding; in particular, \widehat{X} is quasiffine.*
- (iii) *The complement $\overline{X} \setminus p^{-1}(X')$ is small in \overline{X} , and the group H acts freely on the set $p^{-1}(X') \subseteq \widehat{X}$.*
- (iv) *Every H -invariant Weil divisor on the total coordinate space \overline{X} is principal.*

Lemma 2.3. *Suppose that X is a smooth variety, let $\mathfrak{D} \subseteq \text{WDiv}(X)$ be a finitely generated subgroup and consider the sheaf of \mathfrak{D} -graded algebras*

$$\mathcal{S} := \bigoplus_{D \in \mathfrak{D}} \mathcal{S}_D \quad \mathcal{S}_D := \mathcal{O}_X(D).$$

Set $\widetilde{X} := \text{Spec}_X(\mathcal{S})$ and let $q: \widetilde{X} \rightarrow X$ be the canonical morphism. Then, given $D \in \mathfrak{D}$ and a global section $f \in \Gamma(X, \mathcal{O}(D))$, we obtain

$$q^*(D) = \text{div}(f) - q^*(\text{div}(f)),$$

where on the right hand side f is firstly viewed as a homogeneous function on \widetilde{X} , and secondly as a rational function on X . In particular, $q^*(D)$ is principal.

Proof. On suitable open sets $U_i \subseteq X$, we find defining equations f_i^{-1} for D and thus may write $f = h_i f_i$, where $h_i \in \Gamma(U_i, \mathcal{S}_0) = \Gamma(U_i, \mathcal{O})$ and $f_i \in \Gamma(U_i, \mathcal{S}_D)$. Then, on $q^{-1}(U_i)$, we have $q^*(h_i) = h_i$ and the function f_i is homogeneous of degree D and invertible. Thus, we obtain

$$\begin{aligned} q^*(D) &= q^*(\text{div}(f) + D) - q^*(\text{div}(f)) \\ &= q^*(\text{div}(h_i)) - q^*(\text{div}(f)) \\ &= \text{div}(h_i) - q^*(\text{div}(f)) \\ &= \text{div}(h_i f_i) - q^*(\text{div}(f)) \\ &= \text{div}(f) - q^*(\text{div}(f)). \end{aligned}$$

□

Proof of Proposition 2.2. For (i), let $f \in \mathcal{R}(X)^*$ be homogeneous of degree $[D]$. Then its inverse $g \in \mathcal{R}(X)^*$ is homogeneous of degree $-[D]$, and we have $fg \in \mathcal{R}(X)_0^* = \mathbb{K}^*$. According to Remark 2.1, we may view f and g as global sections of sheaves $\mathcal{O}_X(D)$ and $\mathcal{O}_X(-D)$ respectively and thus obtain

$$0 = \text{div}(fg) = (\text{div}(f) + D) + (\text{div}(g) - D).$$

Since the divisors $(\operatorname{div}(f) + D)$ and $(\operatorname{div}(g) - D)$ are both nonnegative, we can conclude $D = \operatorname{div}(f^{-1})$ and hence $[D] = 0$. This implies $f \in \mathcal{R}(X)_0^* = \mathbb{K}^*$ as wanted. To proceed, note that X and X' have the same Cox ring. Thus, normality of $\mathcal{R}(X) = \mathcal{R}(X')$ follows from [3, Prop. 6.3].

We prove most of (iii). For any affine $U \subseteq X$, the set $p^{-1}(U \cap X')$ has the same functions as the affine set $p^{-1}(U)$. Thus, $p^{-1}(U) \setminus p^{-1}(U \cap X')$ must be small. Consequently also $\widehat{X} \setminus p^{-1}(X')$ is small. The fact that H acts freely on $p^{-1}(X')$, is due to the fact that on X' the homogeneous components $\mathcal{R}_{[D]}$ are invertible and hence each H -orbit in $p^{-1}(X')$ admits an invertible function of any degree $w \in K$.

Before proceeding, consider the group $\mathfrak{D} \subseteq \operatorname{WDiv}(X)$ and the associated sheaf \mathcal{S} as used in the definition of \mathcal{R} , and set $\widetilde{X} := \operatorname{Spec}_X(\mathcal{S})$. Then the projection $\mathcal{S} \rightarrow \mathcal{R}$ defines a commutative diagram

$$(2.1) \quad \begin{array}{ccc} \widehat{X} & \xrightarrow{\quad} & \widetilde{X} \\ & \searrow p \quad \swarrow q & \\ & X & \end{array}$$

We turn to (ii). Cover X by affine open subsets $X_f := X \setminus \operatorname{Supp}(D + \operatorname{div}(f))$, where $D \in \mathfrak{D}$ and $f \in \Gamma(X, \mathcal{O}(D))$. Then each f defines an element in $\mathcal{R}(X)$, and hence a function on \widehat{X} and as well on \widetilde{X} . We claim

$$p^{-1}(X_f) = \widehat{X}_f.$$

The inclusion “ \subseteq ” follows from the observation that the function f is invertible on $p^{-1}(X_f)$. Using Lemma 2.3 and the above commutative diagram, we obtain

$$p^{-1}(X_f) \cap p^{-1}(X') = \widehat{X}_f \cap p^{-1}(X').$$

Thus, the complement $\widehat{X}_f \setminus p^{-1}(X_f)$ is small. Since $p^{-1}(X_f)$ is affine, we obtain the desired equality. Consequently, \widehat{X} can be covered by open affine subsets \widehat{X}_f . The corresponding morphisms $\widehat{X}_f \rightarrow \widetilde{X}_f$ are isomorphisms and glue together to the desired open embedding.

Now we easily can finish the proof of (iii). Knowing $\widehat{X} \subseteq \widetilde{X}$ and that both varieties have the same global functions, we obtain that $\widetilde{X} \setminus \widehat{X}$ is small. By what we saw before, then also $\widetilde{X} \setminus p^{-1}(X')$ is small.

We prove (iv). According to (iii), we may assume that X is smooth and we only have to show that every invariant Weil divisor \widehat{D} on \widehat{X} is trivial. Since H acts freely, we have $\widehat{D} = p^*(D)$ with a Weil divisor D on X . Thus, we have to show that all pullback divisors $p^*(D)$ are trivial. For this, it suffices to treat effective divisors D on X , and we may assume that D belongs to \mathfrak{D} . In view of the commutative diagram (2.1), it suffices to show that $q^*(D)$ is principal. This in turn follows immediately from Lemma 2.3. \square

As an important example, we briefly discuss the case of a toric variety, where due to a combinatorial description, Cox ring and universal torsor can be constructed explicitly. First, recall that a toric variety is normal variety Z together with an action of an algebraic torus T and a base point $z_0 \in Z$ such that the orbit map $T \mapsto Z, t \mapsto t \cdot z_0$ is an open embedding.

Any toric variety Z arises from a fan in a lattice N , i.e., a finite collection Σ of strictly convex, polyhedral cones in $N_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} N$ such that for any $\sigma \in \Sigma$ also every face $\sigma_0 \preceq \sigma$ belongs to Σ , and for any two $\sigma_1, \sigma_2 \in \Sigma$ we have $\sigma_1 \cap \sigma_2 \preceq \sigma_i$. The acting torus of Z is $T = \operatorname{Spec}(\mathbb{K}[M])$, where $M := \operatorname{Hom}(N, \mathbb{Z})$ denotes the dual lattice, and Z is the equivariant gluing of the affine toric varieties

$$Z_{\sigma} := \operatorname{Spec}(\mathbb{K}[\sigma^{\vee} \cap M]), \quad \sigma \in \Sigma,$$

where $\sigma^\vee \subseteq M_{\mathbb{Q}}$ is the dual cone, and the T -action on Z_σ is given by the M -grading of the semigroup algebra $\mathbb{K}[\sigma^\vee \cap M]$. There is a bijection from the fan Σ onto the set of T_Z -orbits in Z , sending a cone $\sigma \in \Sigma$ to $\text{orbit}(\sigma)$, the unique closed T -orbit in Z_σ . If $\varrho_1, \dots, \varrho_r$ denote the rays of Σ , then the T -invariant prime divisors of Z are precisely the orbit closures

$$D_Z^i := \overline{\text{orbit}(\varrho_i)}.$$

Now we recall Cox's construction [8]. Suppose that the fan Σ is nondegenerate, i.e., the primitive lattice vectors $v_i \in \varrho_i \cap N$ generate $N_{\mathbb{Q}}$ as a vector space; this just means that we have $\Gamma(Z, \mathcal{O}^*) = \mathbb{K}^*$. Set $F := \mathbb{Z}^r$ and consider the linear map $P: F \rightarrow N$ sending the i -th canonical base vector $e_i \in F$ to $v_i \in N$. There is a fan $\widehat{\Sigma}$ in F consisting of certain faces of the positive orthant $\delta \subseteq F_{\mathbb{Q}}$, namely

$$\widehat{\Sigma} := \{\widehat{\sigma} \preceq \delta; P(\widehat{\sigma}) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

The fan $\widehat{\Sigma}$ defines an open toric subvariety \widehat{Z} of $\overline{Z} = \text{Spec}(\mathbb{K}[\delta^\vee \cap E])$, where $E := \text{Hom}(F, \mathbb{Z})$. As $P: F \rightarrow N$ is a map of the fans $\widehat{\Sigma}$ and Σ , i.e., sends cones of $\widehat{\Sigma}$ into cones of Σ , it defines a morphism $p_Z: \widehat{Z} \rightarrow Z$ of toric varieties. Note that for the unions $W_Z \subseteq Z$ and $W_{\overline{Z}} \subseteq \overline{Z}$ of all at most one codimensional orbits of the respective acting tori, we have $W_{\overline{Z}} = p_Z^{-1}(W_Z) \subseteq \widehat{Z}$.

Let us briefly explain, why $p_Z: \widehat{Z} \rightarrow Z$ is a universal torsor for Z . First, observe that $P: F \rightarrow N$ and its dual map $P^*: M \rightarrow E$ give rise to a pair of exact sequences of abelian groups:

$$\begin{aligned} 0 &\longrightarrow L \xrightarrow{Q^*} F \xrightarrow{P} N \\ 0 &\longleftarrow K \xleftarrow{Q} E \xleftarrow{P^*} M \longleftarrow 0 \end{aligned}$$

The latter sequence has two geometric interpretations. Firstly, it relates the character groups of big tori $T = \text{Spec}(\mathbb{K}[M])$ of Z and $\widehat{T} = \text{Spec}(\mathbb{K}[E])$ of \widehat{Z} to that of $H = \text{Spec}(\mathbb{K}[K])$. Secondly, the lattices M and E represent certain groups of invariant divisors, and K is the divisor class group of Z . Altogether, one has the following commutative diagram.

$$(2.2) \quad \begin{array}{ccccccc} & & \mathbb{X}(H) & \longleftarrow & \mathbb{X}(\widehat{T}) & \longleftarrow & \mathbb{X}(T) \\ & \nearrow & \uparrow \cong & & \uparrow \cong & & \uparrow \cong \\ & & K & \xleftarrow{Q} & E & \xleftarrow{P^*} & M \\ & \nwarrow & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ & & \text{Cl}(Z) & \longleftarrow & \text{WDiv}^T(Z) & \longleftarrow & \text{PDiv}^T(Z) \end{array}$$

$\begin{array}{ccccc} & e \mapsto \chi^e & & u \mapsto \chi^u & \\ & \uparrow \cong & & \uparrow \cong & \\ & & & & \end{array}$
 $\begin{array}{ccccc} & e_i \mapsto D_i & & u \mapsto \text{div}(\chi^u) & \\ & \downarrow \cong & & \downarrow \cong & \\ & & & & \end{array}$

To obtain the Cox ring, consider the groups of divisors $\mathfrak{D} := \text{WDiv}^T(Z)$ and $\mathfrak{D}^0 := \text{PDiv}^T(Z)$, and the sheaf of \mathcal{O}_Z -algebras \mathcal{S} associated to \mathfrak{D} . Then there is a *canonical shifting family*: for any $\text{div}(\chi^u) \in \mathfrak{D}^0$ it sends a section $\chi^{u'}$ to $\chi^{u'-u}$. Let \mathcal{I} denote the associated sheaf of ideals and set $\mathcal{R} := \mathcal{S}/\mathcal{I}$.

Choose any set theoretical section $s: K \rightarrow E$ for $Q: E \rightarrow K$, and, for $w \in K$, let $D(w) \in \text{WDiv}^T(Z)$ denote the divisor corresponding to $s(w)$. Moreover, given a maximal T -invariant affine open subset $W \subseteq Z$, set $\widehat{W} := p_Z^{-1}(W)$. Then, for each $w \in K$ the assignment

$$\Gamma(W, \mathcal{O}_Z(D(w))) \rightarrow \Gamma(\widehat{W}, \mathcal{O}_{\widehat{Z}})_w, \quad \chi^u \mapsto \chi^{(P^*(u) + s(w))}$$

defines an isomorphism of vector spaces. Using this and Remark 2.1, we obtain isomorphisms of sheaves of graded vector spaces, the composition of which is even an isomorphism of sheaves of graded algebras:

$$(2.3) \quad \bigoplus_{w \in K} \mathcal{R}_{[D(w)]} \cong \bigoplus_{w \in K} \mathcal{O}_Z(D(w)) \cong \bigoplus_{w \in K} (p_Z)_*(\mathcal{O}_{\widehat{Z}})_w$$

Passing to the relative spectra, we obtain an H -equivariant isomorphism $\mathrm{Spec}_Z(\mathcal{R}) \cong \widehat{Z}$ over Z and thus see that $p_Z: \widehat{Z} \rightarrow Z$ is a universal torsor. Consequently, the Cox ring of Z is the polynomial ring $\mathcal{O}(\widehat{Z}) = \mathcal{O}(\overline{Z}) = \mathbb{K}[E \cap \delta^\vee]$ with the K -grading given by $\deg(\chi^e) = Q(e)$.

Finally, we note that $H = \mathrm{Spec}(\mathbb{K}[K])$ acts freely on the union $W_{\overline{Z}} \subseteq \overline{Z}$ of all at most one-codimensional torus orbits. Indeed, $W_{\overline{Z}}$ is covered by the affine sets $\overline{Z}_i := \mathrm{Spec}(\mathbb{K}[\varrho_i^\vee \cap E])$, where $\varrho_i = \mathbb{Q}_{\geq 0} \cdot P(e_i)$. Since $P(e_i) = v_i$ is primitive, the images $Q(e_j) \in K$, where $j \neq i$, generate K as an abelian group. Since the corresponding χ^{e_j} are invertible functions on \overline{Z}_i , we may construct invertible $f \in \mathcal{O}(\overline{Z}_i)_w$ for any $w \in K$. This gives freeness of the H -action on $W_{\overline{Z}} \subseteq \overline{Z}$.

Now we use Cox's construction to determine universal torsor and total coordinate space of certain "neatly embedded" subvarieties $X \subseteq Z$ of our toric variety Z . To make this precise, we need the following pullback construction for invariant Weil divisors of Z .

Remark 2.4. Let X be an irreducible variety, which is smooth in codimension one, and let $\varphi: X \rightarrow Z$ a morphism to a toric variety Z such that $\varphi(X)$ intersects the big torus orbit of Z and $X \setminus \varphi^{-1}(W_Z)$ contains no divisors of X . Then the usual pullback of invariant Cartier divisors over W_Z canonically defines a homomorphism

$$\varphi^*: \mathrm{WDiv}^T(Z) \rightarrow \mathrm{WDiv}(X).$$

Note that φ^* takes nonnegative divisors to nonnegative ones. Moreover, one has canonical pullback homomorphisms $\varphi^*\mathcal{O}(D) \rightarrow \mathcal{O}(\varphi^*D)$ for any $D \in \mathrm{WDiv}^T(Z)$. Finally, since $\mathrm{Cl}(Z)$ is obtained as $\mathrm{WDiv}^T(Z)/\mathrm{PDiv}^T(Z)$, there is an induced pull back homomorphism for the divisor class groups

$$\varphi^*: \mathrm{Cl}(Z) \rightarrow \mathrm{Cl}(X).$$

Note that analogous pullback constructions can be performed for any dominant morphism $\varphi: X \rightarrow Y$ of varieties X and Y , which are smooth in codimension one, provided that for the smooth locus $Y' \subseteq Y$ the inverse image $\varphi^{-1}(Y')$ has a small complement in X .

Definition 2.5. Let Z be a toric variety with acting torus T and invariant prime divisors $D_Z^i = \overline{T \cdot z_i}$, where $1 \leq i \leq r$. Moreover, let $X \subseteq Z$ be an irreducible closed subvariety and suppose that X is smooth in codimension one. We call $X \subseteq Z$ a *neat embedding* if

- (i) each $D_X^i := D_Z^i \cap X$ is an irreducible hypersurface in X intersecting the toric orbit $T \cdot z_i$,
- (ii) the pull back homomorphism $\iota^*: \mathrm{Cl}(Z) \rightarrow \mathrm{Cl}(X)$ defined by the inclusion $\iota: X \rightarrow Z$ is an isomorphism.

As we shall see in Section 3 any normal variety with finitely generated Cox ring that admits a closed embedding into a toric variety, admits even a neat embedding into a toric variety.

Now consider a toric variety Z with $\Gamma(Z, \mathcal{O}^*) = \mathbb{K}^*$ and toric universal torsor $p_Z: \widehat{Z} \rightarrow Z$ as constructed before. Let $X \subseteq Z$ be an irreducible closed subvariety. Set $\widehat{X} := p_Z^{-1}(X)$, let $p_X: \widehat{X} \rightarrow X$ denote the restriction of p_Z , and define \overline{X} to be

the closure of \widehat{X} in \overline{Z} . Then everything fits into a commutative diagram:

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{i}} & \overline{Z} \\ \uparrow & & \uparrow \\ \widehat{X} & \xrightarrow{\widehat{i}} & \widehat{Z} \\ \downarrow p_X & & \downarrow p_Z \\ X & \xrightarrow{i} & Z \end{array}$$

where i, \widehat{i} and \overline{i} denote the respective inclusions. The subvarieties $\widehat{X} \subseteq \widehat{Z}$ and $\overline{X} \subseteq \overline{Z}$ are invariant with respect to the action of H on \widehat{Z} and \overline{Z} . In particular, the rings of functions $\Gamma(\mathcal{O}, \overline{X})$ and $\Gamma(\mathcal{O}, \widehat{X})$ are graded by the abelian group $K \cong \text{Cl}(Z)$. Set

$$W_X := X \setminus \bigcup_{i \neq j} D_X^i \cap D_X^j, \quad W_{\overline{X}} := p_X^{-1}(W_X).$$

Theorem 2.6. *Let X be a normal variety and $X \subseteq Z$ neat embedding, suppose that X and Z only have constant globally invertible regular functions and that $X = W_X$ holds. Then there is an isomorphism of K -graded \mathcal{O}_X -algebras*

$$\mathcal{R} \cong (p_X)_* \mathcal{O}_{\widehat{X}},$$

where \mathcal{R} denotes the Cox sheaf on X . Moreover, the variety $\widehat{X} = W_{\overline{X}}$ is normal and the restriction $p_X: \widehat{X} \rightarrow X$ is a universal torsor for X .

Proof. We may assume that $Z = W_Z$ holds. We define the Cox ring of Z as before, using the group $\mathfrak{D}_Z := \text{WDiv}^T(Z)$. Then we have $\mathfrak{D}_Z^0 = \text{PDiv}^T(Z)$ and the associated Cox sheaf $\mathcal{R}_Z = \mathcal{S}_Z/\mathcal{I}_Z$ of $\text{Cl}(Z)$ -graded algebras is built up by means of the canonical shifting family.

Let $\mathfrak{D}_X := i^* \mathfrak{D}_Z$. By the definition of a neat embedding, we have $\mathfrak{D}_X^0 = i^* \mathfrak{D}_Z^0$, and we may define the Cox sheaf $\mathcal{R}_X = \mathcal{S}_X/\mathcal{I}_X$ on X via the pullback shifting family. This gives a commutative diagram

$$\begin{array}{ccc} i^* \mathcal{S}_Z & \longrightarrow & i^* \mathcal{R}_Z \\ \downarrow & & \downarrow \\ \mathcal{S}_X & \longrightarrow & \mathcal{R}_X \end{array}$$

Note that the downwards arrows of the above diagram are even isomorphisms, because by $Z = W_Z$ all divisors of \mathfrak{D}_Z are locally principal. Now, choose any set theoretical section $s: \text{Cl}(Z) \rightarrow \mathfrak{D}$. Then the isomorphisms (2.3) define a commutative diagram providing the desired isomorphism:

$$\begin{array}{ccccc} i^* \mathcal{R}_Z & \xrightarrow{\cong} & i^* \bigoplus \mathcal{O}_Z(s([D])) & \xrightarrow{\cong} & i^* (p_Z)_* \mathcal{O}_{\widehat{Z}} \\ \cong \downarrow & & \downarrow & & \downarrow \cong \\ \mathcal{R}_X & \longrightarrow & \bigoplus \mathcal{O}_X(i^* s([D])) & \longrightarrow & (p_X)_* \mathcal{O}_{\widehat{X}} \end{array}$$

Finally, to see that \widehat{X} is normal, note that the group H acts freely on $\widehat{Z} = W_{\overline{Z}}$. By Luna's slice theorem, \widehat{X} looks locally, in etale topology, like $H \times X$. Since X is normal, and normality is preserved under etale maps, we obtain that \widehat{X} is normal. \square

We give two applications. Recall that a variety X is said to be \mathbb{Q} -factorial if it is normal and for any Weil divisor D on X some positive multiple nD is Cartier. A toric variety Z is \mathbb{Q} -factorial if and only if it arises from a simplicial fan Σ , i.e., each cone of Σ is generated by a linearly independent set of vectors. A toric variety Z with Cox construction $\widehat{Z} \rightarrow Z$ is \mathbb{Q} -factorial if and only if $\widehat{Z} \rightarrow Z$ is a geometric quotient, i.e., its fibers are precisely the H -orbits.

Corollary 2.7. *Let $X \subseteq Z$ be a neat embedding, where X is irreducible and smooth in codimension one and Z is a \mathbb{Q} -factorial toric variety having only constant globally invertible functions.*

- (i) *Suppose that $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ holds and that \widehat{X} is normal. Then X is normal, $p_X: \widehat{X} \rightarrow X$ is a universal torsor for X , and the Cox ring of X is given by*

$$\mathcal{R}(X) = \bigoplus_{w \in K} \Gamma(X, \mathcal{R})_{[t^* D(w)]} \cong \bigoplus_{w \in K} \Gamma(\widehat{X}, \mathcal{O})_w = \Gamma(\widehat{X}, \mathcal{O}).$$

- (ii) *Let W_X be normal with $\Gamma(W_X, \mathcal{O}^*) = \mathbb{K}^*$ and let X^{nor} be the normalization of X . If $\overline{X} \setminus \widehat{X}$ is small, then the Cox ring $\mathcal{R}(X^{\text{nor}})$ is finitely generated and X^{nor} has the H -equivariant normalization $\overline{X}^{\text{nor}}$ of \overline{X} as its total coordinate space.*

Proof. Since Z is \mathbb{Q} -factorial, each fiber of $p_Z: \widehat{Z} \rightarrow Z$ consists of a single H -orbit, having an at most finite isotropy group. Consequently, the map $p_X: \widehat{X} \rightarrow X$ is equidimensional. Since $X \setminus W_X$ is small in X , we can conclude that $\widehat{X} \setminus W_{\widehat{X}}$ is small in \widehat{X} .

We prove (i). Since \widehat{X} is normal, the quotient space $X = \widehat{X} // H$ is normal as well. Moreover, by the above consideration, the isomorphism of K -graded sheaves of Theorem 2.6 extends from W_X to the whole X and the assertion follows.

We turn to (ii). First note that we have $W_X \subseteq X^{\text{nor}}$ and that the complement $X^{\text{nor}} \setminus W_X$ is small. Since $\overline{X} \setminus \widehat{X}$ is small in \overline{X} , the above consideration gives that $\overline{X} \setminus W_{\overline{X}}$ is small in \overline{X} . Consequently, $\overline{X}^{\text{nor}} \setminus W_{\overline{X}}$ is small in $\overline{X}^{\text{nor}}$. The assertion then follows from

$$\mathcal{R}(X^{\text{nor}}) = \Gamma(X^{\text{nor}}, \mathcal{R}) = \Gamma(W_X, \mathcal{R}) = \Gamma(W_{\overline{X}}, \mathcal{O}) = \Gamma(\overline{X}^{\text{nor}}, \mathcal{O}).$$

□

Corollary 2.8. *Let $X \subseteq Z$ be a neat embedding of a normal variety in a smooth toric variety Z , and suppose that X as well as Z only admit constant globally invertible functions. Then $p_X: \widehat{X} \rightarrow X$ is a universal torsor for X , and the Cox ring of X is given by*

$$\mathcal{R}(X) = \bigoplus_{w \in K} \Gamma(X, \mathcal{R})_{[t^* D(w)]} \cong \bigoplus_{w \in K} \Gamma(\widehat{X}, \mathcal{O})_w = \Gamma(\widehat{X}, \mathcal{O}).$$

Moreover, if $\overline{X} \setminus \widehat{X}$ contains no divisors of \overline{X} , then $\mathcal{R}(X)$ is finitely generated and X has the H -equivariant normalization of \overline{X} as its total coordinate space.

3. BUNCHED RINGS AND THEIR VARIETIES

In this section, we generalize the construction of varieties with a prescribed finitely generated Cox ring provided in [5] to the case of divisor class groups with torsion, and we present first basic features of this generalized construction. We begin with discussing the class of rings, which we will consider; the concept 3.1 as well as statements 3.2 and 3.3 are basically due to I.V. Arzhantsev, see [1].

Definition 3.1. Let K be a finitely generated abelian group and $R = \bigoplus_{w \in K} R_w$ any K -graded integral \mathbb{K} -algebra with $R^* = \mathbb{K}^*$.

- (i) We say that a nonzero nonunit $f \in R$ is K -prime if it is homogeneous and $f|gh$ with homogeneous $g, h \in R$ always implies $f|g$ or $f|h$.
- (ii) We say that an ideal $\mathfrak{a} \subset R$ is K -prime if it is homogeneous and for any two homogeneous $f, g \in R$ with $fg \in \mathfrak{a}$ one has $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.
- (iii) We say that a homogeneous prime ideal $\mathfrak{a} \subset R$ has K -height d if d is maximal admitting a chain $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_d = \mathfrak{a}$ of K -prime ideals.
- (iv) We say that the ring R is *factorially graded* if every K -prime ideal of K -height one is principal.

We will briefly indicate the geometric meaning of these notions. By an H -prime divisor on an H -variety we mean a sum $\sum a_D D$ with prime divisors D such that always $a_D = 0, 1$ holds and the D with $a_D = 1$ are transitively permuted by the group H . Note that every H -invariant divisor is a unique sum of H -prime divisors.

Proposition 3.2. Let K be a finitely generated abelian group, R a K -graded normal affine \mathbb{K} -algebra with $R^* = \mathbb{K}^*$, and consider the action of $H := \text{Spec}(\mathbb{K}[K])$ on $\overline{X} := \text{Spec}(R)$. Then the following statements are equivalent.

- (i) The ring R is factorially graded.
- (ii) Every invariant Weil divisor of \overline{X} is principal.
- (iii) Every homogeneous $0 \neq f \in R \setminus R^*$ is a product of K -primes.

Moreover, if one of these statements holds, then a homogeneous nonzero nonunit $f \in R$ is K -prime if and only if $\text{div}(f)$ is H -prime, and every H -prime divisor is of the form $\text{div}(f)$ with a K -prime $f \in R$.

Proof. Assume that (i) holds, and let D be an invariant Weil divisor on \overline{X} . Write $D = D_1 + \cdots + D_r$ with H -prime divisors D_i . Then the vanishing ideal \mathfrak{a}_i of D_i is K -prime of K -height one and thus of the form $\mathfrak{a}_i = \langle f_i \rangle$. We obtain $D_i = \text{div}(f_i)$ and thus $D = \text{div}(f_1 \cdots f_r)$, which verifies (ii).

Assume that (ii) holds. Given a homogeneous $0 \neq f \in R \setminus R^*$, write $\text{div}(f) = D_1 + \cdots + D_r$ with H -prime divisors D_i . Then we have $D_i = \text{div}(f_i)$, and one directly verifies that the f_i are homogeneous K -primes. This gives a decomposition $f = a \cdot f_1 \cdots f_r$ with some $a \in R^* = \mathbb{K}^*$ as required in (i).

If (iii) holds and \mathfrak{a} is a K -prime ideal of height one, then we take any homogeneous $0 \neq f \in \mathfrak{a}$ and find a K -prime factor f_1 of f with $f_1 \in \mathfrak{a}$. Thus, we have inclusions $0 \subset \langle f_1 \rangle \subseteq \mathfrak{a}$ of K -prime ideals, which implies $\mathfrak{a} = \langle f_1 \rangle$, verifying (i). \square

Remark 3.3. A normal factorially K -graded affine \mathbb{K} -algebra R with a free finitely generated abelian group K corresponds to a normal affine variety \overline{X} with an action of a torus H . In this case, every Weil divisor on \overline{X} is H -linearizable and hence linear equivalent to an H -invariant one and thus principal, i.e., R is even factorial. However, if K has torsion, then there may exist normal affine algebras which are factorially K -graded but not factorial, see [1, Example 4.2].

For a finitely generated abelian group K , we denote by $K^t \subseteq K$ the torsion part, set $K^0 := K/K^t$ and write $w^0 \in K^0$ for the class of $w \in K$. Given a free abelian group E and a homomorphism $Q: E \rightarrow K$, we denote by $Q^0: E_{\mathbb{Q}} \rightarrow K_{\mathbb{Q}}^0$ the induced linear map sending $a \otimes e$ to $a \otimes Q(e)^0$. Moreover, the relative interior of a cone $\sigma \subseteq V$ in a rational vector space V is denoted by σ° .

Definition 3.4. Consider a finitely generated abelian group K and a normal factorially K -graded affine \mathbb{K} -algebra R . Let

$$\mathfrak{F} = \{f_1, \dots, f_r\} \subset R$$

be a system of homogeneous pairwise non associated K -prime generators for R . The *projected cone associated to* $\mathfrak{F} \subset R$ is

$$(E \xrightarrow{Q} K, \gamma),$$

where $E := \mathbb{Z}^r$, the homomorphism $Q: E \rightarrow K$ sends $e_i \in E$ to $w_i := \deg(f_i) \in K$ and $\gamma \subseteq E_{\mathbb{Q}}$ is the convex cone generated by e_1, \dots, e_r .

- (i) We say that $\mathfrak{F} \subset R$ is *admissible*, if, for each facet $\gamma_0 \preceq \gamma$, the image $Q(\gamma_0 \cap E)$ generates the abelian group K .
- (ii) A face $\gamma_0 \preceq \gamma$ is called an *\mathfrak{F} -face* if the product over all f_i with $e_i \in \gamma_0$ does not belong to the ideal $\sqrt{\langle f_j; e_j \notin \gamma_0 \rangle} \subseteq R$.
- (iii) An *\mathfrak{F} -bunch* is a nonempty collection Φ of projected \mathfrak{F} -faces $Q^0(\gamma_0) \subseteq K_{\mathbb{Q}}$ with the following properties:
 - (a) a projected \mathfrak{F} -face $\tau \subseteq K_{\mathbb{Q}}$ belongs to Φ if and only if for each $\tau \neq \sigma \in \Phi$ we have $\emptyset \neq \tau^\circ \cap \sigma^\circ \neq \sigma^\circ$,
 - (b) for each facet $\gamma_0 \prec \gamma$, there is a cone $\tau \in \Phi$ such that $\tau^\circ \subseteq Q^0(\gamma_0)^\circ$ holds.

A *bunched ring* is a triple (R, \mathfrak{F}, Φ) , where $\mathfrak{F} \subset R$ is an admissible system of generators and Φ is an \mathfrak{F} -bunch.

Construction 3.5. Let (R, \mathfrak{F}, Φ) be a bunched ring and $(E \xrightarrow{Q} K, \gamma)$ its projected cone. The associated *collection of relevant faces* and *covering collection* are

$$\begin{aligned} \text{rlv}(\Phi) &:= \{ \gamma_0 \preceq \gamma; \gamma_0 \text{ an } \mathfrak{F}\text{-face with } \tau^\circ \subseteq Q^0(\gamma_0)^\circ \text{ for some } \tau \in \Phi \}, \\ \text{cov}(\Phi) &:= \{ \gamma_0 \in \text{rlv}(\Phi); \gamma_0 \text{ minimal} \}. \end{aligned}$$

Consider the action of the diagonalizable group $H := \text{Spec}(\mathbb{K}[K])$ on $\overline{X} := \text{Spec}(R)$, and the open H -invariant subsets of \overline{X} defined by

$$\widehat{X}(R, \mathfrak{F}, \Phi) := \bigcup_{\gamma_0 \in \text{rlv}(\Phi)} \overline{X}_{\gamma_0}, \quad \overline{X}_{\gamma_0} := \overline{X}_{f_1^{u_1} \dots f_r^{u_r}} \text{ for some } (u_1, \dots, u_r) \in \gamma_0^\circ.$$

Then the H -action on $\widehat{X} := \widehat{X}(R, \mathfrak{F}, \Phi)$ admits a good quotient; we denote the quotient variety by

$$X := X(R, \mathfrak{F}, \Phi) := \widehat{X}(R, \mathfrak{F}, \Phi) // H.$$

The subsets $\overline{X}_{\gamma_0} \subseteq \widehat{X}$, where $\gamma_0 \in \text{rlv}(\Phi)$, are saturated with respect to the quotient map $p_X: \widehat{X} \rightarrow X$, and X is covered by the affine open subsets $X_{\gamma_0} := p_X(\overline{X}_{\gamma_0})$.

Before proving the statements made in this construction, we present its basic features. Recall that an *A_2 -variety* is a variety, in which any two points admit a common affine neighbourhood.

Theorem 3.6. *Let (R, \mathfrak{F}, Φ) be a bunched ring with projected cone $(E \xrightarrow{Q} K, \gamma)$, and denote $\widehat{X} := \widehat{X}(R, \mathfrak{F}, \Phi)$ and $X := X(R, \mathfrak{F}, \Phi)$. Then X is a normal A_2 -variety with*

$$\dim(X) = \dim(R) - \dim(K_{\mathbb{Q}}^0), \quad \Gamma(X, \mathcal{O}^*) = \mathbb{K}^*,$$

and X admits a neat embedding into a toric variety. Moreover, there is a canonical isomorphism $\text{Cl}(X) \cong K$, the Cox ring $\mathcal{R}(X)$ is isomorphic to R and $p_X: \widehat{X} \rightarrow X$ is a universal torsor for X .

We come to the proofs of the assertions. The key is a combinatorial understanding of the variation of good quotients for diagonalizable group actions on affine varieties. So, let K be a finitely generated abelian group, R a K -graded affine \mathbb{K} -algebra, and consider the action of $H := \text{Spec}(\mathbb{K}[K])$ on $\overline{X} := \text{Spec}(R)$.

We say that an open subset $W \subseteq \overline{X}$ is *$(H, 2)$ -maximal* if it admits a good quotient $W \rightarrow W // H$ with an A_2 -variety $W // H$ and there is no $W' \subseteq \overline{X}$ with the same

property comprising W as a proper subset and saturated under $W' \rightarrow W'//H$. Our task is to describe the $(H, 2)$ -maximal sets. We define the *weight cone* $\omega_H(\overline{X}) \subseteq K_{\mathbb{Q}}^0$ and, for any $z \in \overline{X}$, its *orbit cone* $\omega_H(z) \subseteq K_{\mathbb{Q}}^0$ as

$$\begin{aligned}\omega_H(\overline{X}) &:= \text{cone}(w^0; w \in K \text{ with } R_w \neq 0), \\ \omega_H(z) &:= \text{cone}(w^0; w \in K \text{ with } f(z) \neq 0 \text{ for some } f \in R_w).\end{aligned}$$

Remark 3.7. For a bunched ring (R, \mathfrak{F}, Φ) with projected cone $(E \xrightarrow{Q} K, \gamma)$, the weight cone of the action of $H = \text{Spec}(\mathbb{K}[K])$ on $\overline{X} = \text{Spec}(R)$ is $\omega_H(\overline{X}) = Q^0(\gamma)$, and the orbit cones are precisely the cones $Q^0(\gamma_0)$, where $\gamma_0 \preceq \gamma$ is an \mathfrak{F} -face.

The orbit cone of a point $z \in \overline{X}$ describes the set of H -orbits in the orbit closure $\overline{H \cdot z}$. Note that this set becomes partially ordered by writing $H \cdot z_1 < H \cdot z_0$ if $H \cdot z_1$ is contained in the closure of $H \cdot z_0$.

Proposition 3.8. *For any point $z \in \overline{X}$, there is an order preserving bijection of finite sets*

$$\begin{aligned}\{H\text{-orbits in } \overline{H \cdot z}\} &\longrightarrow \text{faces}(\omega_H(z)), \\ H \cdot z_0 &\mapsto \omega_H(z_0),\end{aligned}$$

and the inverse map is obtained as follows: given $\omega_0 \preceq \omega_H(z)$, take any $w^0 \in \omega_0^\circ$ admitting an $f \in R_w$ with $f(z) \neq 0$ and assign the unique closed orbit $H \cdot z_0$ in $\overline{H \cdot z} \cap \overline{X}_f$ to ω_0 .

Proof. Split H as $H = H^0 \times G$ with the component of identity $H^0 = \text{Spec}(\mathbb{K}[K^0])$ and a finite abelian group G . The H^0 acts on \overline{X}/G such that the quotient map $\pi: \overline{X} \rightarrow \overline{X}/G$ becomes H^0 -equivariant. For any $z \in \overline{X}$, the orbit cones $\omega_H(z)$ and $\omega_{H^0}(\pi(z))$ coincide, and we have $H \cdot z_0 \subseteq H \cdot z$ if and only if $H^0 \cdot \pi(z_0) \subseteq H^0 \cdot \pi(z)$. Thus, we may assume that $H = H^0$ holds, i.e. that H is a torus. Now observe that it suffices to prove the assertions in the case $\overline{X} = \overline{H \cdot z}$. But then they are basic facts on (not necessarily normal) affine toric varieties. \square

The set $\Omega_H(\overline{X})$ of all orbit cones is finite. By a *2-maximal* collection of orbit cones, we mean a subset $\Psi \subseteq \Omega_H(\overline{X})$ being maximal with the property that for any two $\omega_1, \omega_2 \in \Psi$ one has $\omega_1^\circ \cap \omega_2^\circ \neq \emptyset$. Moreover, for any $w \in K$, we define its *GIT-cone* to be the convex polyhedral cone

$$\lambda(w) := \bigcap_{w^0 \in \omega \in \Omega_H(\overline{X})} \omega \subseteq K_{\mathbb{Q}}^0.$$

Proposition 3.9. *Let K be a finitely generated abelian group, R a factorially K -graded affine \mathbb{K} -algebra, and consider the action of $H := \text{Spec}(\mathbb{K}[K])$ on $\overline{X} := \text{Spec}(R)$. Then there are mutually inverse bijections*

$$\begin{aligned}\{2\text{-maximal collections } \in \Omega_H(\overline{X})\} &\longleftrightarrow \{(H, 2)\text{-maximal subsets of } \overline{X}\}, \\ \Psi &\mapsto \{z \in \overline{X}; \omega_0 \preceq \omega(z) \text{ for some } \omega_0 \in \Psi\}, \\ \{\omega(z); H \cdot z \text{ closed in } W\} &\longleftrightarrow W.\end{aligned}$$

Moreover, the collection of GIT-cones $\Lambda(\overline{X}) = \{\lambda(w); w \in K\}$ is a fan in $K_{\mathbb{Q}}^0$ having the weight cone $\omega_H(\overline{X})$ as its support, and there is a canonical injection

$$\begin{aligned}\Lambda &\longrightarrow \{2\text{-maximal collections } \in \Omega_H(\overline{X})\}, \\ \lambda &\mapsto \{\omega \in \Omega_H(\overline{X}); \lambda^\circ \subseteq \omega^\circ\}\end{aligned}$$

The $(H, 2)$ -maximal open subset of $\widehat{X}(\lambda) \subseteq \overline{X}$ arising in this way from $\lambda \in \Lambda(\overline{X})$ is the set of semistable points of any $w \in K$ with $w^0 \in \lambda^\circ$:

$$\widehat{X}(\lambda) = \overline{X}^{\text{ss}}(w) := \bigcup_{f \in \oplus_{n>0} R_{nw}} \overline{X}_f = \{z \in \overline{X}; \lambda \subseteq \omega(z)\}.$$

In particular, the $(H, 2)$ -maximal subsets arising from the $\lambda \in \Lambda$ are precisely those having a quasiprojective quotient space. Moreover, one has

$$\lambda \preceq \lambda' \iff \widehat{X}(\lambda) \supseteq \widehat{X}(\lambda').$$

Finally, the quotient spaces $\widehat{X}(\lambda) // H$ are all projective over $\text{Spec}(R_0)$; in particular, they are projective if and only if $R_0 = \mathbb{K}$ holds.

The fan $\Lambda(\overline{X})$ of all GIT-cones is called the *GIT-fan* of the H -variety \overline{X} , and its cones are also referred to as *chambers*.

Proof of Proposition 3.9. As earlier, choose a splitting $H = H^0 \times G$ into the component of identity $H^0 = \text{Spec}(\mathbb{K}[K^0])$ and a finite abelian subgroup $G \subseteq H$. Then \overline{X}/G comes with an induced H^0 -action making the quotient map $\pi: \overline{X} \rightarrow \overline{X}/G$ equivariant. The $(H^0, 2)$ -maximal subsets of \overline{X}/G are in bijection with the $(H, 2)$ -maximal subsets of \overline{X} via $V \mapsto \pi^{-1}(V)$. Moreover, for any $z \in \overline{X}$, the orbit cones $\omega_H(z)$ and $\omega_{H^0}(\pi(z))$ coincide.

By these considerations it suffices to prove the assertion for the action of H^0 on \overline{X}/G . This is done in [2, Sec. 1]. Note that the assumption of factoriality posed in [2, Prop. 1.8] can be weakened to requiring that for every H^0 -invariant Weil divisor D on \overline{X}/G some multiple is principal, which in turn is easily verified by considering the pull back divisor $\pi^*(D)$ on \overline{X} . \square

Lemma 3.10. *In the notation of Construction 3.5, the open subset $\widehat{X} \subseteq \overline{X}$ equals the $(H, 2)$ -maximal open subset $W(\Psi) \subseteq \overline{X}$ defined by the 2-maximal collection*

$$\Psi := \{\omega \in \Omega_H(\overline{X}); \tau^\circ \subseteq \omega^\circ \text{ for some } \tau \in \Phi\} = \{Q^0(\gamma_0); \gamma_0 \in \text{rlv}(\Phi)\}.$$

Moreover, an orbit $H \cdot z \subseteq \widehat{X}$ is closed in \widehat{X} if and only if we have $\omega_H(z) = Q^0(\gamma_0)$ for some $\gamma_0 \in \text{rlv}(\Phi)$.

Proof. Let $z \in \overline{X}_{\gamma_0}$ for some $\gamma_0 \in \text{rlv}(\Phi)$. Since $\overline{X}_{\gamma_0} = \overline{X}_{f_1^{u_1} \dots f_r^{u_r}}$ does not change under variation of $u \in \gamma_0^\circ$, we obtain $Q^0(\gamma_0) \subseteq \omega(z)$. Thus, $Q^0(\gamma_0)^\circ \subseteq \omega(z_0)^\circ$ holds for some z_0 in the closure of $H \cdot z$. By 2-maximality, we have $\omega(z_0) \in \Psi$. This implies $z \in W(\Psi)$. Conversely, given $z \in W(\Psi)$, the orbit cone $\omega(z)$ comprises some $\tau \in \Phi$. Let z_0 be a point in the closure of $H \cdot z$ in \overline{X} with $\tau^\circ \subseteq \omega_H(z_0)^\circ$. Then $\gamma_0 := \text{cone}(e_i; f_i(z_0) \neq 0)$ is a relevant face with $z \in \overline{X}_{\gamma_0}$.

Finally, let $H \cdot z$ be any orbit in \widehat{X} . If $H \cdot z$ is closed in \widehat{X} , then we have $\omega_H(z) \in \Psi$ and hence $\omega_H(z) = Q^0(\gamma_0)$ for some $\gamma_0 \in \text{rlv}(\Phi)$. Conversely, let $\omega_H(z) = Q^0(\gamma_0)$ hold for some $\gamma_0 \in \text{rlv}(\Phi)$. Then we have $\omega_H(z) \in \Psi$. If $H \cdot z$ were not closed in \widehat{X} , we had $\omega_H(z_0) \prec \omega_H(z)$ with some $z_0 \in \widehat{X}$ such that $H \cdot z_0$ is closed in \widehat{X} , which implies $\omega_H(z_0) \in \Psi$. Thus, we obtain $\omega_H(z_0)^\circ \cap \omega_H(z)^\circ \neq \emptyset$, a contradiction. \square

Proof of Construction 3.5 and Theorem 3.6, part 1. By Lemma 3.10 and Proposition 3.9, the good quotient space $X = X(R, \mathfrak{F}, \Phi)$ exists and is an A_2 -variety. It inherits normality from \widehat{X} , and we have

$$\Gamma(X, \mathcal{O}^*) = \Gamma(\widehat{X}, \mathcal{O}^*)^H = \mathbb{K}^*.$$

In order to verify the assertion on the dimension, it suffices to show that the generic fiber of $p_X: \widehat{X} \rightarrow X$ is a single H -orbit of dimension $\dim(H)$. For this, note the generic orbit cone equals $Q^0(\gamma)$. In particular, the generic orbits are of full dimension. Property 3.4 (iii) of a bunched ring guarantees $Q^0(\gamma) \in \text{rlv}(\Phi)$. Lemma 3.10 then says that the generic orbits are closed in \widehat{X} and thus they show up as fibers of $p_X: \widehat{X} \rightarrow X$.

Finally, we have to show that each $\overline{X}_{\gamma_0} \subseteq \widehat{X}$, where $\gamma_0 \in \text{rlv}(\Phi)$, is saturated with respect to the quotient map $p_X: \widehat{X} \rightarrow X$. For this, it suffices to show that given

any $\gamma_0 \in \text{rlv}(\Phi)$ and $H \cdot z \subseteq \overline{X}_{\gamma_0}$, which is closed in \overline{X}_{γ_0} , then $H \cdot z$ is already closed in \widehat{X} . Since $H \cdot z \subseteq \widehat{X}$ holds, there must be a face $\omega_0 \preceq \omega_H(z)$ with $\omega_0 \in \text{rlv}(\Phi)$, see Lemma 3.10. Thus, we have

$$\omega_0^\circ \cap Q^0(\gamma_0)^\circ \neq \emptyset.$$

Moreover, denoting by $\overline{H \cdot z}$ the closure of $H \cdot z$ in \overline{X} , Proposition 3.8 tells us that the orbit $H \cdot z_0 \subseteq \overline{H \cdot z}$ corresponding to $\omega_0 \preceq \omega_H(z)$ is the unique closed orbit in $\overline{H \cdot z} \cap \overline{X}_{f_1^{u_1} \dots f_r^{u_r}}$. This implies $H \cdot z_0 = H \cdot z$ and hence $\omega_H(z) \in \text{rlv}(\Phi)$. Lemma 3.10 thus ensures that $H \cdot z$ is closed in \widehat{X} . \square

The next step is to provide an explicit isomorphism $K \rightarrow \text{Cl}(X)$ for the variety $X = X(R, \mathfrak{F}, \Phi)$. We need the following preparation.

Construction 3.11. Let (R, \mathfrak{F}, Φ) be a bunched ring with $(E \xrightarrow{\mathcal{Q}} K, \gamma)$ as its projected cone, consider the action of $H = \text{Spec}(\mathbb{K}[K])$ on $\overline{X} = \text{Spec}(R)$ and set

$$\widehat{X} := \widehat{X}(R, \mathfrak{F}, \Phi), \quad X := X(R, \mathfrak{F}, \Phi).$$

Then, for every member $f_i \in \mathfrak{F}$ its zero set $V(\widehat{X}, f_i)$ is an H -prime divisor and thus its image $D_X^i \subseteq X$ is a prime divisor. Moreover, by 3.4 (iii), part (b), we have

$$W_{\overline{X}} := \overline{X} \setminus \bigcup_{i \neq j} V(\overline{X}; f_i, f_j) = \bigcup_{\gamma_0 \text{ facet of } \gamma} \overline{X}_{\gamma_0} \subseteq \widehat{X},$$

and hence the complement $\overline{X} \setminus W_{\overline{X}}$ is small in \overline{X} . By 3.4 (i), the group H acts freely on $W_{\overline{X}}$. Moreover, as a union of saturated subsets $W_{\overline{X}}$ is saturated under $p_X: \widehat{X} \rightarrow X$. Its image $W_X := p_X(W_{\overline{X}})$ is open with small complement in X and is given by

$$W_X = X \setminus \bigcup_{i \neq j} D_X^i \cap D_X^j.$$

Proposition 3.12. *In the notation of 3.11, cover $W_{\overline{X}}$ by H -invariant open subsets W_j admitting w -homogeneous functions $h_j \in \Gamma(\mathcal{O}^*, W_j)$. For every $w \in K$, fix a w -homogeneous function $h_w \in \mathbb{K}(\widehat{X})^*$.*

- (i) *For any $w \in K$ there is a unique divisor $D(h_w)$ on X with $p_X^*(D(h_w)) = \text{div}(h_w)$; on each $p_X(W_j)$ it is given by $D(h_w) = \text{div}(h_w/h_j)$.*
- (ii) *Let $U \subseteq X$ be open. Then, for every $w \in K$, there is an isomorphism of \mathbb{K} -vector spaces*

$$\Gamma(U, \mathcal{O}(D(h_w))) \rightarrow \Gamma(p_X^{-1}(U), \mathcal{O})_w, \quad g \mapsto p_X^*(g)h_w.$$

- (iii) *The assignment $w \mapsto D(h_w)$ induces an isomorphism from K onto $\text{Cl}(X)$, not depending on the choice of h_w :*

$$\overline{D}_X: K \rightarrow \text{Cl}(X), \quad w \mapsto \overline{D}(w) := [D(h_w)].$$

- (iv) *For $h_w = f_i \in \mathfrak{F}$, we obtain $D(f_i) = D_X^i$. In particular, $\text{Cl}(X)$ is generated by the classes $\overline{D}(w_i)$ of the D_X^i , where $1 \leq i \leq r$ and $w_i = \deg(f_i)$.*

Proof. The H -invariant local equations h_w/h_j define a Cartier divisor on $W_X = p_X(W_{\overline{X}})$. Since W_X has a small complement in X , this Cartier divisor extends in a unique manner to a Weil divisor $D(h_w)$ on X . This shows (i).

For (ii), note that on each $U_j := p_X(W_j) \cap U$ the section g is given as $g = g'_j h_j/h_w$ with a regular function $g'_j \in \mathcal{O}(U_j)$. Consequently, the function $p_X^*(g)h_w$ is regular on $W_{\overline{X}} \cap p_X^{-1}(U)$, and thus, by normality, on $p_X^{-1}(U)$. In particular, the assignment is a well defined homomorphism. Moreover, $f \mapsto f/h_w$ defines an inverse homomorphism.

We turn to (iii). To see that the class $\overline{D}(w)$ does not depend on the choice of h_w , consider a further w -homogeneous $g_w \in \mathbb{K}(X)^*$. Then $f := h_w/g_w$ is an invariant rational function descending to X , where we obtain

$$D(h_w) - D(g_w) = \operatorname{div}(f).$$

Thus, $\overline{D}: K \rightarrow \operatorname{Cl}(X)$ is well defined and, by construction, it is homomorphic. To verify injectivity, let $D(h_w) = \operatorname{div}(f)$ for some $f \in \mathbb{K}(X)^*$. Then we obtain $\operatorname{div}(h_w) = \operatorname{div}(p_X^*(f))$. Thus, $h_w/p_X^*(f)$ is an invertible function on \overline{X} . By $R^* = \mathbb{K}^*$, it must be constant. This implies $w = 0$. For surjectivity, let any $D \in \operatorname{WDiv}(X)$ be given. Then $p_X^*(D)$ is an invariant divisor on \overline{X} . Since R is factorially graded, we obtain $p_X^*(D) = \operatorname{div}(f)$ with some rational function h on \overline{X} , which is homogeneous of some degree w . This means $D = D(h)$.

Finally, for (iv), we only have to show that D_X^i equals $D(f_i)$. By construction, these divisors have the same support, and we have $p_X^*(D(f_i)) = \operatorname{div}(f_i)$. Since $\operatorname{div}(f_i)$ is H -prime and D_X^i is prime, the assertion follows. \square

We indicate now, how to embed the variety X arising from a given bunched ring (R, \mathfrak{F}, Φ) into a toric variety. Let $(E \xrightarrow{Q} K, \gamma)$ be the associated projected cone. Then, with $M := \ker(Q)$, we have mutually dual exact sequences of rational vector spaces

$$0 \longrightarrow L_{\mathbb{Q}} \longrightarrow F_{\mathbb{Q}} \xrightarrow{P} N_{\mathbb{Q}} \longrightarrow 0$$

$$0 \longleftarrow K_{\mathbb{Q}}^0 \xleftarrow{Q^0} E_{\mathbb{Q}} \longleftarrow M_{\mathbb{Q}} \longleftarrow 0$$

Recall that $\gamma = \operatorname{cone}(e_1, \dots, e_r)$ holds with the canonical basis e_1, \dots, e_r for the lattice $E = \mathbb{Z}^r$. We denote the dual basis in $F = \operatorname{Hom}(E, \mathbb{Z})$ again by e_1, \dots, e_r . Then $\delta := \operatorname{cone}(e_1, \dots, e_r) \subseteq F_{\mathbb{Q}}$ is the dual cone of $\gamma \subseteq E_{\mathbb{Q}}$.

In order to relate bunches of cones to fans and vice versa, we need to transform collections of faces of γ to collections of faces of δ and vice versa. This is done element-wise via the face correspondence

$$\begin{aligned} \operatorname{faces}(\delta) &\longleftrightarrow \operatorname{faces}(\gamma) \\ \delta_0 &\mapsto \delta_0^* := \delta_0^\perp \cap \gamma \\ \gamma_0^\perp \cap \delta =: \gamma_0^* &\longleftarrow \gamma_0 \end{aligned}$$

Given any two corresponding collections $\widehat{\mathfrak{S}} \subseteq \operatorname{faces}(\delta)$ and $\widehat{\Phi} \subseteq \operatorname{faces}(\gamma)$, we infer from [4, Lemma 4.3] that the following two compatibility statements are equivalent.

- (i) Any two $\widehat{\sigma}_1, \widehat{\sigma}_2 \in \widehat{\mathfrak{S}}$ admit an L -invariant separating linear form.
- (ii) For any two $\widehat{\tau}_1, \widehat{\tau}_2 \in \widehat{\Phi}$ one has $Q^0(\widehat{\tau}_1)^\circ \cap Q^0(\widehat{\tau}_2)^\circ \neq \emptyset$.

Given a compatible collection $\widehat{\mathfrak{S}} \subseteq \operatorname{faces}(\delta)$, the images $P(\widehat{\sigma})$, where $\widehat{\sigma} \in \widehat{\mathfrak{S}}$ generate a fan Σ in $N_{\mathbb{Q}}$, which we call the *image fan* of $\widehat{\mathfrak{S}}$. Moreover, we call the fan $\widehat{\Sigma}$ generated by a compatible $\widehat{\mathfrak{S}}$ a *projectable fan* in $F_{\mathbb{Q}}$. Note that for a maximal compatible collection $\widehat{\mathfrak{S}} \subseteq \operatorname{faces}(\delta)$, the image fan consists precisely of the images $P(\widehat{\sigma})$, where $\widehat{\sigma} \in \widehat{\mathfrak{S}}$, whereas $\widehat{\Sigma} = \widehat{\mathfrak{S}}$ need not hold in general.

Construction 3.13. Let (R, \mathfrak{F}, Φ) be a bunched ring and $(E \xrightarrow{Q} K, \gamma)$ its projected cone, consider the action of $H = \operatorname{Spec}(\mathbb{K}[K])$ on $\overline{X} = \operatorname{Spec}(R)$ and set

$$\widehat{X} := \widehat{X}(R, \mathfrak{F}, \Phi), \quad X := X(R, \mathfrak{F}, \Phi).$$

Define a K -grading on the polynomial ring $\mathbb{K}[T_1, \dots, T_r]$ by $\deg(T_i) := \deg(f_i)$. This gives an H -equivariant closed embedding into $\overline{Z} := \mathbb{K}^r$, namely

$$\bar{\iota}: \overline{X} \rightarrow \overline{Z}, \quad z \mapsto (f_1(z), \dots, f_r(z)).$$

Let $\widehat{\Theta} \subseteq \text{faces}(\gamma)$ be any compatible collection with $\text{rlv}(\Phi) \subseteq \widehat{\Theta}$. Then the restriction $\widehat{\iota}$ of $\bar{\iota}$ to \widehat{X} defines a commutative diagram

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{\iota}} & \widehat{Z} \\ \parallel H \downarrow p_X & & p_Z \downarrow \parallel H \\ X & \xrightarrow{\iota} & Z \end{array}$$

with the toric varieties \widehat{Z} and Z defined by the projectable fan $\widehat{\Sigma}$ associated to the collection $\widehat{\Theta}$ corresponding to $\widehat{\Theta}$ and the image fan Σ respectively.

Proposition 3.14. *In the setting of Construction 3.13, the following statements hold.*

- (i) $p_Z: \widehat{Z} \rightarrow Z$ is a toric Cox construction, and we have $\widehat{X} = \bar{\iota}^{-1}(\widehat{Z})$. In particular, $\widehat{\iota}: \widehat{X} \rightarrow \widehat{Z}$ is an H -equivariant closed embedding.
- (ii) For $1 \leq i \leq r$, let $D_Z^i \subseteq Z$ be the invariant prime divisor corresponding to $e_i \in E$. Then $D_X^i = \iota^*(D_Z^i)$ holds, and, as a consequence, $W_X = \iota^{-1}(W_Z)$.
- (iii) The induced morphism $\iota: X \rightarrow Z$ of the quotient varieties is a neat closed embedding.

Proof of Construction 3.13 and Proposition 3.14. In order to see that $p_Z: \widehat{Z} \rightarrow Z$ is a toric Cox construction, we only have to check that the images $P(e_i) \in N$ are pairwise different primitive lattice vectors in N and that the rays through them occur in the image fan Σ . But this is guaranteed by $\text{rlv}(\Phi) \subseteq \widehat{\Theta}$ and the Properties 3.4 (i) and (iii), part (b) of Φ .

By construction, \widehat{Z} is the union of all subsets $\overline{Z}_{\gamma_0} = \overline{Z}_{T^u}$, where $\gamma_0 \in \widehat{\Theta}$ and $u \in \gamma_0^\circ$. Thus $\text{rlv}(\Phi) \subseteq \widehat{\Theta}$ ensures $\widehat{X} \subseteq \bar{\iota}^{-1}(\widehat{Z})$. To show equality, we may assume, by suitably enlarging \widehat{Z} , that $\widehat{\Theta}$ is 2-maximal. Then $\widehat{Z} \subseteq \overline{Z}$ is the associated $(H, 2)$ -maximal set. To see that a given point $z \in \overline{X} \cap \widehat{Z}$ lies in $\bar{\iota}(\widehat{X})$, we may assume that $H \cdot z$ is closed in \widehat{Z} . Then $\omega_H(z)$ belongs to $\widehat{\Theta}$. Moreover, $\omega_H(z)$ is a projected \mathfrak{F} -face, and thus, by 2-maximality, belongs to $\text{rlv}(\Phi)$. This implies $z \in \bar{\iota}(\widehat{X})$.

Together, this proves Construction 3.13 and the first assertion of Proposition 3.14. The second one follows from the commutative diagram of Construction 3.13 and

$$p_X^* D_X^i = \text{div}(f_i) = \widehat{\iota}^* \text{div}(T_i) = \widehat{\iota}^* p_Z^* D_Z^i.$$

To see that $\iota: X \rightarrow Z$ is a neat embedding, we just note that the isomorphisms $\overline{D}_X: K \rightarrow \text{Cl}(X)$ and $\overline{D}_Z: K \rightarrow \text{Cl}(Z)$ of Proposition 3.12 fit into a commutative diagram

$$\begin{array}{ccc} & K & \\ \cong \swarrow & & \searrow \cong \\ \text{Cl}(X) & \xleftarrow{\iota^*} & \text{Cl}(Z) \end{array}$$

□

Corollary 3.15. *If the toric ambient variety Z of Construction 3.13 is \mathbb{Q} -factorial, then $p_Z^{-1}(X) \rightarrow X$ is a universal torsor for X and the closure of $p_Z^{-1}(X)$ in \overline{Z} is the total coordinate space of X .*

Proof of Theorem 3.6, part 2. Proposition 3.12 gives the isomorphism of divisor class groups. Taking $\hat{\Theta} = \text{rlv}(\Phi)$ in 3.13 gives the desired neat embedding $\iota: X \rightarrow Z$. In order to see that $p_X: \hat{X} \rightarrow X$ is a universal torsor, note that, by normality of \hat{X} it suffices to prove this for the restriction $p_X: W_{\overline{X}} \rightarrow W_X$. This in turn follows Theorem 2.6 applied to any neat embedding $\iota: X \rightarrow Z$ as provided by Construction 3.13. Using once more normality, we obtain that R is the Cox ring of X . \square

4. BASIC GEOMETRIC PROPERTIES

In this section, we indicate how to read off basic geometric properties of the variety associated to a bunched ring from its defining data. We treat an explicit example, and at the end of the section, we characterize the class of varieties that arise from bunched rings. Many statements are direct generalizations of those of [5]. However our proofs are different, as we mostly don't use neat embeddings.

In the sequel, (R, \mathfrak{F}, Φ) is a bunched ring with projected cone $(E \xrightarrow{Q} K, \gamma)$ and admissible system of generators $\mathfrak{F} = \{f_1, \dots, f_r\}$. We consider the action of $H := \text{Spec}(\mathbb{K}[K])$ on $\overline{X} := \text{Spec}(R)$ and set

$$\hat{X} := \hat{X}(R, \mathfrak{F}, \Phi), \quad X := X(R, \mathfrak{F}, \Phi).$$

Recall from Proposition 3.12 that we associated to any $h_w \in Q(R)_w^*$ a divisor $D(h_w)$ and this defines a canonical isomorphism $\overline{D}_X: K \rightarrow \text{Cl}(X)$. In particular, we may canonically identify $K_{\mathbb{Q}}^0 \cong \mathbb{Q} \otimes K$ with the rational divisor class group $\text{Cl}_{\mathbb{Q}}(X)$.

In our first statement, we describe some cones of divisors. Recall that a divisor on a variety is called *movable* if it has a positive multiple with base locus of codimension at least two and it is called *semiample* if it has a base point free multiple.

Proposition 4.1. *The cones of effective, movable, semiample and ample divisor classes of X in $\text{Cl}_{\mathbb{Q}}(X) = K_{\mathbb{Q}}^0$ are given as*

$$\text{Eff}(X) = Q^0(\gamma), \quad \text{Mov}(X) = \bigcap_{\gamma_0 \text{ facet of } \gamma} Q^0(\gamma_0),$$

$$\text{Sample}(X) = \bigcap_{\tau \in \Phi} \tau, \quad \text{Ample}(X) = \bigcap_{\tau \in \Phi} \tau^{\circ}.$$

Lemma 4.2. *In the setting of Proposition 3.12, consider the divisor $D(h_w)$ on X , a section $g \in \Gamma(X, \mathcal{O}(D(h_w)))$, the sets*

$$Z(g) := \text{Supp}(\text{div}(g) + D(h_w)), \quad X_g := X \setminus Z(g),$$

and let $f := p_X^(g)h_w \in \Gamma(\overline{X}, \mathcal{O})_w$ denote the homogeneous function corresponding to g . Then we have*

$$Z(g) = p_X(V(\hat{X}; f)), \quad p_X^{-1}(Z(g) \cap W_X) = V(\overline{X}; f) \cap W_{\overline{X}}.$$

Moreover, if the open subset $X_g \subseteq X$ is affine, then its inverse image is given as $\pi_X^{-1}(X_g) = \overline{X}_f$.

Proof. For the first equation, by surjectivity of p_X , it suffices to show $p_X^{-1}(X_g) = W$, where

$$W := \hat{X} \setminus p_X^{-1}(p_X(V(\hat{X}, f))).$$

Consider $g^{-1} \in \Gamma(X_g, \mathcal{O}(D(h_w^{-1})))$. The corresponding function $f' = p_X^*(g^{-1})h_w^{-1}$ on $p_X^{-1}(X_g)$ satisfies $f'f = 1$, which implies $p_X^{-1}(X_g) \subseteq W$. Now consider $f^{-1} \in \Gamma(W, \mathcal{O})_w$. The corresponding section $g' = f^{-1}h_w$ in $\Gamma(\pi(W), \mathcal{O}(D(h_w^{-1})))$ satisfies $g'g = 1$, which implies $\pi(W) \subseteq X_g$.

The second equation follows directly from the first one and the fact that on $W_{\overline{X}} = p_X^{-1}(W_X)$ the p_X -fibers are precisely the H -orbits and hence $V(\overline{X}; f) \cap W_{\overline{X}}$ is saturated with respect to p_X .

Finally, let X_g be affine. The first equation gives us $p_X^{-1}(X_g) \subseteq \overline{X}_f$. Moreover, $p_X^{-1}(X_g)$ is affine, and thus its boundary $\overline{X} \setminus p_X^{-1}(X_g)$ is of pure codimension one. Inside $W_{\overline{X}}$, this boundary coincides with $V(\overline{X}; f)$. Since $\overline{X} \setminus W_{\overline{X}}$ is small, the assertion follows. \square

Proof of Proposition 4.1. We have to characterize in terms of $w_0 \in K_{\mathbb{Q}}^0$, when a divisor $D(h_w) \in \text{WDiv}(X)$ as constructed in Proposition 3.12 is effective, movable, semiample or ample. The description of the effective cone $\text{Eff}(X)$ is clear by Proposition 3.12 (ii).

For the description of the moving cone $\text{Mov}(X)$, we may replace X with W_X . By Proposition 3.12 (ii) and Lemma 4.2, the base locus of $D(h_w)$ has a component of codimension one if and only if there is an $f_i \in \mathfrak{F}$ dividing all $f \in R_w$. Thus, $D(h_w)$ has small stable base locus if and only if we have

$$w^0 \in \bigcap_{i=1}^r \text{cone}(w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_r).$$

For the description of the semiample cone, note that by Lemma 4.2 the divisor $D(h_w)$ is semiample if and only if \widehat{X} is contained in the union of all \overline{X}_f , where $f \in R_{nw}$ and $n > 0$. This in turn is equivalent to

$$w^0 \in \bigcap_{z \in \widehat{X}} \omega(z) = \bigcap_{\tau \in \Phi} \tau.$$

We come to the ample cone. Suppose first that $w^0 \in \tau^\circ$ holds for all $\tau \in \Phi$. Then, for some $n > 0$, we may write \widehat{X} as the union of p_X -saturated open subsets \overline{X}_f with functions $f \in R_{nw}$. For the corresponding sections $g \in \Gamma(X; \mathcal{O}(D(h_w^n)))$, Lemma 4.2 gives $p_X(\overline{X}_f) = X_g$. In particular, all X_g are affine and thus $D(h_w)$ is ample.

Now suppose that $D(h_w)$ is ample. We may assume that $D(h_w)$ is even very ample. Then, for every $f^u = f_1^{u_1} \cdots f_r^{u_r}$ with $Q(u) = w$ the corresponding section $g^u \in \Gamma(X; \mathcal{O}(D(h_w)))$ defines an affine set $X_{g^u} \subseteq X$, and these sets cover X . By Lemma 4.2, we obtain saturated inclusions $\overline{X}_{f^u} \subseteq \widehat{X}$ and hence a saturated inclusion $\widehat{X}(w) \subseteq \widehat{X}$, see Proposition 3.9 for the notation. By $(H, 2)$ -maximality of $\widehat{X}(w)$, we obtain $\widehat{X}(w) = \widehat{X}$ and hence $w \in \tau^\circ$ for every $\tau \in \Phi$. \square

Corollary 4.3. *If we vary the \mathfrak{F} -bunch Φ , then the ample cones of the quasiprojective ones among the resulting varieties $X = X(R, \mathfrak{F}, \Phi)$ are precisely the cones*

$$\lambda^\circ \subseteq \bigcap_{\gamma_0 \text{ facet of } \gamma} Q^0(\gamma_0)^\circ$$

with $\lambda \in \Lambda(\overline{X})$. If $\text{SAmple}(X) \preceq \text{SAmple}(X')$ holds, then $\widehat{X}' \subseteq \widehat{X}$ induces a projective morphism $X' \rightarrow X$, which is an isomorphism in codimension one.

In order to investigate local properties of X , we first observe that X comes by construction with a decomposition into locally closed pieces.

Construction 4.4. To any \mathfrak{F} -face $\gamma_0 \preceq \gamma$, we associate a locally closed subset, namely

$$\overline{X}(\gamma_0) := \{z \in \overline{X}; f_i(z) \neq 0 \Leftrightarrow e_i \in \gamma_0\} \subseteq \overline{X}.$$

These sets are pairwise disjoint and cover the whole \overline{X} . Taking the pieces defined by relevant \mathfrak{F} -faces, one obtains a constructible subset

$$\tilde{X} := \bigcup_{\gamma_0 \in \text{rlv}(\Phi)} \overline{X}(\gamma_0) \subseteq \hat{X},$$

which is precisely the union of all closed H -orbits of \hat{X} . The images of the pieces inside \tilde{X} form a decomposition of X into pairwise disjoint locally closed pieces:

$$X = \bigcup_{\gamma_0 \in \text{rlv}(\Phi)} X(\gamma_0), \quad \text{where } X(\gamma_0) := p_X(\overline{X}(\gamma_0)).$$

Example 4.5. If we have $R = \mathbb{K}[T_1, \dots, T_r]$ and $\mathfrak{F} = \{T_1, \dots, T_r\}$, then X is the toric variety arising from the image fan Σ associated to $\text{rlv}(\Phi)$, and for any $\gamma_0 \in \text{rlv}(\Phi)$, the piece $X(\gamma_0) \subseteq X$ is precisely the toric orbit corresponding to the cone $P(\gamma_0^*) \in \Sigma$.

Proposition 4.6. *For any $\gamma_0 \in \text{rlv}(\Phi)$, the associated piece $X(\gamma_0)$ of the decomposition 4.4 has the following descriptions.*

- (i) *For any neat embedding $X \subseteq Z$ as constructed in 3.13, the piece $X(\gamma_0)$ is the intersection of X with the toric orbit of Z corresponding to γ_0 .*
- (ii) *In terms of the prime divisors $D_X^i \subseteq X$ defined by the generators $f_i \in \mathfrak{F}$, the piece $X(\gamma_0)$ is given as*

$$X(\gamma_0) = \bigcap_{e_i \notin \gamma_0} D_X^i \setminus \bigcup_{e_j \in \gamma_0} D_X^j$$

- (iii) *In terms of the open subsets $X_{\gamma_i} \subseteq X$ and $\overline{X}_{\gamma_i} \subseteq \overline{X}$ defined by the relevant faces $\gamma_i \in \text{rlv}(\Phi)$, we have*

$$\begin{aligned} X(\gamma_0) &= X_{\gamma_0} \setminus \bigcup_{\gamma_0 \prec \gamma_1 \in \text{rlv}(\Phi)} X_{\gamma_1}, \\ p_X^{-1}(X(\gamma_0)) &= \overline{X}_{\gamma_0} \setminus \bigcup_{\gamma_0 \prec \gamma_1 \in \text{rlv}(\Phi)} \overline{X}_{\gamma_1}. \end{aligned}$$

Proof of Construction 4.4 and Proposition 4.6. Obviously, \overline{X} is the union of the locally closed $\overline{X}(\gamma_0)$, where $\gamma_0 \preceq \gamma$ runs through the \mathfrak{F} -faces. To proceed, recall from Lemma 3.10 and Proposition 3.9 that $\hat{X} \subseteq \overline{X}$ is the $(H, 2)$ -maximal subset given by the 2-maximal collection

$$\Psi = \{Q^0(\gamma_0); \gamma_0 \in \text{rlv}(\Phi)\} = \{\omega(z); H \cdot z \text{ closed in } \hat{X}\}.$$

Given $z \in \hat{X}$, we have $\omega_H(z) = Q^0(\gamma_0) \in \Psi$. Thus, Lemma 3.10 tells us that $H \cdot z$ is closed in \hat{X} . Conversely, if $H \cdot z$ is closed in \hat{X} , consider the \mathfrak{F} -face

$$\gamma_0 := \text{cone}(e_i; f_i(z) \neq 0) \preceq \gamma.$$

Then we have $z \in \overline{X}(\gamma_0)$ and $Q^0(\gamma_0) = \omega_H(z) \in \Psi$; see Lemma 3.10. Hence γ_0 is a relevant face. This implies $z \in \hat{X}$.

All further statements are most easily seen by means of a neat embedding $X \subseteq Z$ as constructed in 3.13. Let $\overline{X} \subseteq \overline{Z} = \mathbb{K}^r$ denote the closed H -equivariant embedding arising from \mathfrak{F} . Then, \overline{X} intersects precisely the $\overline{Z}(\gamma_0)$, where γ_0 is an \mathfrak{F} -face, and in these cases we have

$$\overline{X}(\gamma_0) = \overline{Z}(\gamma_0) \cap \overline{X}.$$

As mentioned in Example 4.5, the images $Z(\gamma_0) = p_Z(\overline{Z}(\gamma_0))$, where $\gamma_0 \in \text{rlv}(\Phi)$, are precisely the toric orbits of Z . Moreover, we have

$$\hat{X} = \hat{Z} \cap \overline{X}, \quad \tilde{X} = \tilde{Z} \cap \overline{X}.$$

Since p_Z separates H -orbits along \tilde{Z} , we obtain $X(\gamma_0) = Z(\gamma_0) \cap X$ for every $\gamma_0 \in \text{rlv}(\Phi)$. Consequently, the $X(\gamma_0)$, where $\gamma_0 \in \text{rlv}(\Phi)$, are pairwise disjoint and form a decomposition of X into locally closed pieces. Finally, using $X(\gamma_0) = Z(\gamma_0) \cap X$ and $D_X^i = \iota^*(D_Z^i)$, we obtain Assertions 4.6 (ii) and (iii) directly from the corresponding representations of the toric orbit $Z(\gamma_0)$. \square

We say that a divisor class $[D] \in \text{Cl}(X)$ is Cartier at a point $x \in X$ if some representative D is principal near x . Similarly, we say that $[D] \in \text{Cl}(X)$ is \mathbb{Q} -Cartier at x if some nonnegative multiple of a representative D is principal near x .

Proposition 4.7. *Consider a relevant face $\gamma_0 \in \text{rlv}(\Phi)$, a point $x \in X(\gamma_0)$, and let $w \in K$.*

- (i) *The class $\overline{D}_X(w)$ is Cartier at x if and only if $w \in Q(\text{lin}(\gamma_0) \cap E)$ holds.*
- (ii) *The class $\overline{D}_X(w)$ is \mathbb{Q} -Cartier at x if and only if $w^0 \in \text{lin}(Q^0(\gamma_0))$ holds.*

Lemma 4.8. *Consider a relevant face $\gamma_0 \in \text{rlv}(\Phi)$, a point $z \in \overline{X}(\gamma_0)$. Then, for any $w \in K$, the following statements are equivalent.*

- (i) *There is an invertible homogeneous function $f \in \Gamma(H \cdot z, \mathcal{O})_w$.*
- (ii) *One has $w \in Q(\text{lin}(\gamma_0) \cap E)$.*

Proof. We have to determine the set $K_z \subseteq K$ of degrees $w \in K$ admitting an invertible homogeneous function on $H \cdot z$. Recall that the orbit $H \cdot z$ is closed in \overline{X}_{γ_0} , and thus the homogeneous functions on $H \cdot z$ are precisely the restrictions of the homogeneous functions on \overline{X}_{γ_0} . Moreover, we have $z \in \overline{X}(\gamma_0)$, and thus $f_i(z) \neq 0$ if and only if $e_i \in \gamma_0$. Thus, the invertible homogenous functions on $H \cdot z$ are generated by the products $f_1^{u_1} \cdots f_r^{u_r}$, where $u \in \text{lin}((\gamma_0) \cap E)$. Consequently, we obtain $K_z = Q(\text{lin}((\gamma_0) \cap E)) \subseteq K$. \square

Proof of Proposition 4.7. The divisor $D := D(h_w)$ is Cartier at x if and only if there is a neighbourhood $U \subseteq X$ of x and an invertible section $g \in \Gamma(U, \mathcal{O}(D))$. By Proposition 3.12, the latter is equivalent to the existence of an invertible homogeneous function $f \in \Gamma(H \cdot z, \mathcal{O})_w$ on the closed orbit $H \cdot z \subseteq p_X^{-1}(x)$. Now, Lemma 4.8 gives the desired equivalence, and the first statement is proven. The second is an immediate consequence of the first. \square

Corollary 4.9. *Inside the divisor class group $\text{Cl}(X) \cong K$, the Picard group of X is given by*

$$\text{Pic}(X) = \bigcap_{\gamma_0 \in \text{cov}(\Phi)} Q(\text{lin}(\gamma_0) \cap E).$$

Proof. A divisor $\overline{D}(w)$ is Cartier if and only if it is Cartier along any $X(\gamma_0)$, where $\gamma_0 \in \text{rlv}(\Phi)$. Since we have $\text{cov}(\Phi) \subseteq \text{rlv}(\Phi)$, and for any $\gamma_0 \in \text{rlv}(\Phi)$, there is a $\gamma_1 \in \text{cov}(\Phi)$ with $\gamma_1 \preceq \gamma_0$, it suffices to take the intersection over $\text{cov}(\Phi)$. \square

A point $x \in X$ is factorial (\mathbb{Q} -factorial) if and only if every Weil divisor is Cartier (\mathbb{Q} -Cartier) at x . Thus, Proposition 4.7 has the following application to singularities.

Corollary 4.10. *Consider a relevant face $\gamma_0 \in \text{rlv}(\Phi)$ and point $x \in X(\gamma_0)$.*

- (i) *The point x is factorial if and only if Q maps $\text{lin}(\gamma_0) \cap E$ onto K .*
- (ii) *The point x is \mathbb{Q} -factorial if and only if $Q^0(\gamma_0)$ is of full dimension.*

Corollary 4.11. *The variety X is \mathbb{Q} -factorial if and only if Φ consists of cones of full dimension.*

Whereas local factoriality admits a simple combinatorial characterization, smoothness is difficult in general. Nevertheless, we have the following statement.

Proposition 4.12. *Suppose that \widehat{X} is smooth, let $\gamma_0 \in \text{rlv}(\Phi)$, and $x \in X(\gamma_0)$. Then x is a smooth point if and only if Q maps $\text{lin}(\gamma_0) \cap E$ onto K .*

Proof. The “only if” part is clear by Corollary 4.10. Conversely, if Q maps $\text{lin}(\gamma_0) \cap E$ onto K , then Lemma 4.8 says that the fibre $p_X^{-1}(x)$ consists of a single free H -orbit. Consequently, H acts freely over an open neighbourhood $U \subseteq X$ of x . Thus x is smooth. \square

Corollary 4.13. *Let $X \subseteq Z$ be a neat embedding into a toric variety Z as constructed in 3.13, and let $x \in X$.*

- (i) *The point x is a factorial (\mathbb{Q} -factorial) point of X if and only if it is a smooth (\mathbb{Q} -factorial) point of Z .*
- (ii) *If \widehat{X} is smooth, then x is a smooth point of X if and only if it is a smooth point of Z .*

As an immediate consequence of general results on quotient singularities, see [6] and [10], one obtains the following.

Proposition 4.14. *Suppose that \widehat{X} is smooth. Then X has at most rational singularities. In particular, X is Cohen-Macaulay.*

In the case of a complete intersection $\overline{X} \subseteq \overline{Z}$, we obtain a simple description of the canonical divisor class; the proof given in [5, Theorem 9.1] works without changes, and therefore is omitted.

Proposition 4.15. *Suppose that the relations of \mathfrak{F} are generated by K -homogeneous polynomials $g_1, \dots, g_d \in \mathbb{K}[T_1, \dots, T_r]$, where $d := r - \text{rank}(K) - \dim(X)$. Then, in $K \cong \text{Cl}(X)$, the canonical divisor class of X is given by*

$$D_X^c = \sum_{j=1}^d \deg(g_j) - \sum_{i=1}^r \deg(f_i).$$

A variety is called (\mathbb{Q}) -Gorenstein if (some multiple of) its anticanonical divisor is Cartier. Moreover, it is called (\mathbb{Q}) -Fano if (some multiple of) its anticanonical class is an ample Cartier divisor.

Corollary 4.16. *In the setting of Proposition 4.15, the following statements hold.*

- (i) *X is \mathbb{Q} -Gorenstein if and only if*

$$\sum_{i=1}^r \deg(f_i) - \sum_{j=1}^d \deg(g_j) \in \bigcap_{\tau \in \Phi} \text{lin}(\tau),$$

- (ii) *X is Gorenstein if and only if*

$$\sum_{i=1}^r \deg(f_i) - \sum_{j=1}^d \deg(g_j) \in \bigcap_{\gamma_0 \in \text{cov}(\Phi)} Q(\text{lin}(\gamma_0) \cap E).$$

- (iii) *X is \mathbb{Q} -Fano if and only if we have*

$$\sum_{i=1}^r \deg(f_i) - \sum_{j=1}^d \deg(g_j) \in \bigcap_{\tau \in \Phi} \tau^\circ,$$

- (iv) *X is Fano if and only if we have*

$$\sum_{i=1}^r \deg(f_i) - \sum_{j=1}^d \deg(g_j) \in \bigcap_{\tau \in \Phi} \tau^\circ \cap \bigcap_{\gamma_0 \in \text{cov}(\Phi)} Q(\text{lin}(\gamma_0) \cap E).$$

Example 4.17. Set $K := \mathbb{Z}^2$ and consider the K -grading of $\mathbb{K}[T_1, \dots, T_5]$ defined by $\deg(T_i) := \deg(f_i)$, where w_i is the i -th column of the matrix

$$Q := \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}$$

Then this K -grading descends to a K -grading of the following factorial residue algebra

$$R := \mathbb{K}[T_1, \dots, T_5] / \langle T_1T_2 + T_3^2 + T_4T_5 \rangle.$$

The classes $f_i \in R$ of $T_i \in \mathbb{K}[T_1, \dots, T_5]$, where $1 \leq i \leq 5$, form a system $\mathfrak{F} \subset R$ of pairwise nonassociated homogeneous prime generators of R and

$$\Phi := \{\tau\}, \quad \tau := \text{cone}(w_2, w_5)$$

defines an \mathfrak{F} -bunch with associated projected cone $(E \xrightarrow{Q} K_0, \gamma)$, where $E = \mathbb{Z}^5$, and $\gamma = \text{cone}(e_1, \dots, e_5)$. The covering collection $\text{cov}(\Phi)$ consists of

$$\text{cone}(e_1, e_4), \quad \text{cone}(e_2, e_5), \quad \text{cone}(e_1, e_2, e_3), \quad \text{cone}(e_3, e_4, e_5).$$

The corresponding variety $X = X(R, \mathfrak{F}, \Phi)$ is a \mathbb{Q} -factorial projective surface. It has a single singularity, namely the point in the piece $X(\gamma_0)$ for $\gamma_0 = \text{cone}(e_2, e_5)$. The Picard group of X is of index 3 in $\text{Cl}(X)$, the canonical class of X is Cartier and X is Fano.

Example 4.18. Let $K := \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, and consider the K -grading of the polynomial ring $\mathbb{K}[T_1, \dots, T_6]$ given by

$$\deg(T_1) = (1, \bar{1}), \quad \deg(T_2) = (1, \bar{2}), \quad \deg(T_3) = (1, \bar{1}),$$

$$\deg(T_4) = (1, \bar{2}), \quad \deg(T_5) = (1, \bar{1}), \quad \deg(T_6) = (1, \bar{2}).$$

Similarly as in the preceding example, this K -grading descends to a K -grading of the factorial residue algebra

$$R := \mathbb{K}[T_1, \dots, T_6] / \langle T_1T_2 + T_3T_4 + T_5T_6 \rangle.$$

The classes $f_i \in R$ of $T_i \in \mathbb{K}[T_1, \dots, T_6]$, where $1 \leq i \leq 6$, form a system $\mathfrak{F} \subset R$ of pairwise nonassociated homogeneous prime generators of R and $\Phi := \{\mathbb{Q}_{\geq 0}\}$ defines an \mathfrak{F} -bunch with associated projected cone $(E \xrightarrow{Q} K_0, \gamma)$ with $E = \mathbb{Z}^6$ and $K_0 = \mathbb{Z}$, where Q sends e_i and we have $\gamma = \text{cone}(e_1, \dots, e_6)$.

The resulting variety $X = X(R, \mathfrak{F}, \Phi)$ is \mathbb{Q} -factorial, projective, of dimension four and has divisor class group $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Its singular locus is of dimension two; it comprises, for example, the surface $X(\gamma_0)$ for $\gamma_0 = \text{cone}(e_1, e_3, e_5)$. The Picard group of X is free cyclic and sits in the divisor class group of X as

$$(3 \cdot \mathbb{Z}, \bar{0}) \subseteq \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

Finally, we figure out the class of varieties that arise from bunched rings. We say that a variety X is A_2 -maximal if it is A_2 and admits no open embedding $X \subsetneq X'$ into an A_2 -variety X' such that $X' \setminus X$ is small in X' . For example, every projective variety is A_2 -maximal.

Theorem 4.19. *Let X be a normal variety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group $\text{Cl}(X)$ and finitely generated Cox ring $\mathcal{R}(X)$. Then the following statements are equivalent.*

- (i) $X \cong X(R, \mathfrak{F}, \Phi)$ holds with some bunched ring (R, \mathfrak{F}, Φ) .
- (ii) X is A_2 -maximal and every $\text{Cl}(X)^0$ -homogeneous $f \in \mathcal{R}(X)^*$ is constant.

Proof. Suppose that X is as in (ii). Set $K := \text{Cl}(X)$ and $R := \mathcal{R}(X)$. Then any $f \in R^*$ is necessarily K^0 -homogeneous and hence by assumption constant. Moreover, by Proposition 2.2, the Cox ring R is factorially K -graded. Consider the total coordinate space $\overline{X} = \text{Spec}(R)$ with its action of $H = \text{Spec}(\mathbb{K}[K])$. Then the universal torsor $p_X: \widehat{X} \rightarrow X$ is a good quotient for the action of H .

Since X is A_2 -maximal, the subset $\widehat{X} \subseteq \overline{X}$ must be $(H, 2)$ -maximal. Thus \widehat{X} arises from a 2-maximal collection Ψ of orbit cones. Choose any system \mathfrak{F} of R of pairwise nonassociated K -prime generators. By Proposition 2.2, we have a small complement $\overline{X} \setminus p_X^{-1}(X')$, where $X' \subseteq X$ denotes the set of smooth points, and H acts freely on $p_X^{-1}(X')$. This ensures that \mathfrak{F} is admissible. Denoting by Φ the collection of minimal members of Ψ , we obtain the desired bunched ring (R, \mathfrak{F}, Φ) .

Now suppose that (i) holds, i.e., let $X = X(R, \mathfrak{F}, \Phi)$. By construction, X is an A_2 -variety with finitely generated Cox ring. Thus, we only have to show that X is A_2 -maximal. For this, suppose we have an open embedding $X \subseteq X'$ with small boundary into an A_2 -variety. Replacing, if necessary, X' with its normalization, we may assume that X' is normal. Then X and X' share the same Cox ring R and they occur as good quotients of open subsets $\widehat{X} \subseteq \widehat{X}'$ of their common total coordinate space. By H -maximality of \widehat{X} , we obtain $\widehat{X} = \widehat{X}'$ and thus $X = X'$. \square

Corollary 4.20. *Let X be a normal A_2 -maximal variety with $\Gamma(X, \mathcal{O}) = \mathbb{K}$ and finitely generated Cox ring. Then $X \cong X(R, \mathfrak{F}, \Phi)$ holds with some bunched ring (R, \mathfrak{F}, Φ) .*

Corollary 4.21. *Let X be a normal projective variety with finitely generated Cox ring. Then $X \cong X(R, \mathfrak{F}, \Phi)$ holds with some bunched ring (R, \mathfrak{F}, Φ) .*

5. AMBIENT MODIFICATION

Here, we investigate the behaviour of Cox rings under modifications induced from toric modifications. As a preparation, we first consider modifications $Z_1 \rightarrow Z_0$ of toric varieties, their lifting properties to the respective Cox constructions, and give an explicit description of the situation in terms of coordinates.

Let Σ_0 be a fan in a lattice N , and let $v_1, \dots, v_r \in N$ denote the primitive generators of the rays of Σ_0 . Suppose that v_1, \dots, v_r generate $N_{\mathbb{Q}}$ as a vector space and, for some $2 \leq d \leq r$, we have

$$\sigma_0 := \text{cone}(v_1, \dots, v_d) \in \Sigma_0.$$

Definition 5.1. Let $v_{\infty} \in \sigma_0^{\circ}$ be a primitive lattice vector. We define the *index* of v_{∞} in σ_0 to be the minimal number $m_{\infty} \in \mathbb{Z}_{\geq 1}$ such that there are nonnegative integers a_1, \dots, a_d with

$$m_{\infty} v_{\infty} = a_1 v_1 + \dots + a_d v_d.$$

The star of σ_0 in Σ_0 is defined as $\text{star}_{\Sigma_0}(\sigma_0) := \{\sigma \in \Sigma_0; \sigma_0 \preceq \sigma\}$, and the stellar subdivision of Σ_0 at a primitive lattice vector $v_{\infty} \in \sigma_0^{\circ}$ is the fan

$$\Sigma_1 := (\Sigma_0 \setminus \text{star}_{\Sigma_0}(\sigma_0)) \cup \{\tau + \text{cone}(v_{\infty}); \tau \not\preceq \sigma \in \text{star}_{\Sigma_0}(\sigma_0)\}.$$

We will consider the Cox constructions $P_0: \widehat{\Sigma}_0 \rightarrow \Sigma_0$ and $P_1: \widehat{\Sigma}_1 \rightarrow \Sigma_1$. Recall that the fans $\widehat{\Sigma}_0$ and $\widehat{\Sigma}_1$ live in the lattices

$$F_0 := \bigoplus_{i=1}^r \mathbb{Z}e_i, \quad F_1 := \bigoplus_{i=1}^r \mathbb{Z}e_i \oplus \mathbb{Z}e_{\infty}$$

and consist of certain faces of the positive orthants $\delta_0 \subseteq (F_0)_{\mathbb{Q}}$ and $\delta_1 \subseteq (F_1)_{\mathbb{Q}}$. More precisely, as lattice homomorphisms, the projection maps are given by

$$P_0: F_0 \rightarrow N, \quad e_i \mapsto v_i, \quad P_1: F_1 \rightarrow N, \quad e_i \mapsto v_i, \quad e_{\infty} \mapsto v_{\infty};$$

note that in order to have $e_\infty \mapsto v_\infty$ in the Cox construction of Σ_1 , it is necessary that $v_\infty \in N$ is a primitive lattice vector. The fans $\widehat{\Sigma}_0$ and $\widehat{\Sigma}_1$ are given by

$$\widehat{\Sigma}_0 = \{\delta'_0 \preceq \delta_0; P_0(\delta'_0) \subseteq \sigma \in \Sigma_0\}, \quad \widehat{\Sigma}_1 = \{\delta'_1 \preceq \delta_1; P_1(\delta'_1) \subseteq \sigma \in \Sigma_1\}.$$

Now, fix nonnegative integers a_1, \dots, a_d as in Definition 5.1, and define lattice homomorphisms

$$\begin{aligned} G: F_1 &\rightarrow F_0, & e_i &\mapsto e_i, & e_\infty &\mapsto a_1 e_1 + \dots + a_d e_d, \\ G_1: F_1 &\rightarrow F_1, & e_i &\mapsto e_i, & e_\infty &\mapsto m_\infty e_\infty. \end{aligned}$$

Then all the lattice homomorphisms defined so far fit into the following commutative diagram

$$\begin{array}{ccc} & F_1 & \\ G_1 \swarrow & & \searrow G \\ F_1 & & F_0 \\ P_1 \downarrow & & \downarrow P_0 \\ N & \xrightarrow{\text{id}} & N \end{array}$$

Note that G defines a map of the fans $\widehat{\Sigma}_1$ and $\widehat{\Sigma}_0$ and, as well, of the respective fans of faces $\overline{\Sigma}_1$ and $\overline{\Sigma}_0$ of the positive orthants δ_1 and δ_0 .

Proposition 5.2. *Let Z_i, \widehat{Z}_i and \overline{Z}_i be the toric varieties associated to the fans $\Sigma_i, \widehat{\Sigma}_i$ and $\overline{\Sigma}_i$ respectively. Then the maps of the fans just defined give rise to a commutative diagram of toric morphisms*

$$\begin{array}{ccccc} & & \overline{Z}_1 & & \\ & \swarrow \overline{\pi}_1 & \uparrow & \searrow \overline{\pi} & \\ \overline{Z}_1 & & \widehat{Z}_1 & & \overline{Z}_0 \\ & \nwarrow \widehat{\pi}_1 & \downarrow \widehat{\pi} & \nearrow \widehat{\pi} & \\ \widehat{Z}_1 & & \widehat{Z}_0 & & \\ p_1 \downarrow & & \downarrow p_0 & & \\ Z_1 & \xrightarrow{\pi} & Z_0 & & \end{array}$$

where $\pi: Z_1 \rightarrow Z_0$ properly contracts an invariant prime divisor, $p_i: \widehat{Z}_i \rightarrow Z_i$ are universal torsors, $\overline{\pi}: \overline{Z}_1 \rightarrow \overline{Z}_0$ is the quotient for a \mathbb{K}^* -action and $\overline{\pi}_1: \overline{Z}_1 \rightarrow \overline{Z}_1$ is the quotient for an action of the group C_{m_∞} of m_∞ -th roots of the unity.

As announced, we will also look at this in terms of coordinates. For this, recall that, given a point y of any affine \mathbb{K}^* -variety Y , one says that

- the limit $\lim_{t \rightarrow 0} t \cdot y$ exists, if the morphism $\mathbb{K}^* \rightarrow Y, t \mapsto t \cdot y$ admits a continuation to \mathbb{K} ,
- the limit $\lim_{t \rightarrow \infty} t \cdot y$ exists, if the morphism $\mathbb{K}^* \rightarrow Y, t \mapsto t^{-1} \cdot y$ admits a continuation to \mathbb{K} .

Moreover, one defines the *plus* and *minus cells* of the affine \mathbb{K}^* -variety Y to be the closed subsets

$$Y^+ := \{y \in Y; \lim_{t \rightarrow 0} t \cdot y \text{ exists}\}, \quad Y^- := \{y \in Y; \lim_{t \rightarrow \infty} t \cdot y \text{ exists}\}.$$

The fixed point set $Y^0 \subseteq Y$ is given by $Y^0 = Y^+ \cap Y^-$. We denote by $Y^s := Y \setminus (Y^+ \cup Y^-)$ the union of all closed \mathbb{K}^* -orbits.

Lemma 5.3. *Consider the coordinates z_1, \dots, z_r on $\overline{Z}_0 = \mathbb{K}^r$ corresponding to $e_1, \dots, e_r \in F_0$ and $z_1, \dots, z_r, z_\infty$ on $\overline{Z}_1 = \mathbb{K}^{r+1}$ corresponding to $e_1, \dots, e_r, e_\infty \in F_1$.*

(i) *The \mathbb{K}^* -action on \overline{Z}_1 and its good quotient $\overline{\pi}: \overline{Z}_1 \rightarrow \overline{Z}_0$ are given by*

$$\begin{aligned} t \cdot (z_1, \dots, z_r, z_\infty) &= (t^{-a_1} z_1, \dots, t^{-a_d} z_d, z_{d+1}, \dots, z_r, t z_\infty), \\ \overline{\pi}(z_1, \dots, z_r, z_\infty) &= (z_\infty^{a_1} z_1, \dots, z_\infty^{a_d} z_d, z_{d+1}, \dots, z_r). \end{aligned}$$

(ii) *Plus and minus cells and the fixed point set of the \mathbb{K}^* -action are given by*

$$\begin{aligned} \overline{Z}_1^+ &= V(\overline{Z}_1; z_1, \dots, z_d), \\ \overline{Z}_1^- &= V(\overline{Z}_1; z_\infty), \\ \overline{Z}_1^0 &= V(\overline{Z}_1; z_1, \dots, z_d, z_\infty). \end{aligned}$$

In particular, $\overline{Z}_1^- = \overline{p_1^{-1}(E)}$, where $E \subseteq Z_1$ is the exceptional divisor of $\pi: Z_1 \rightarrow Z_0$.

(iii) *The images of \overline{Z}_1^0 and \overline{Z}_1^- under $\overline{\pi}: \overline{Z}_1 \rightarrow \overline{Z}_0$ both equal $V(\overline{Z}_0; z_1, \dots, z_d)$, we have $\overline{Z}_1^+ \subseteq \overline{Z}_1 \setminus \widehat{Z}_1$ and*

$$\overline{Z}_1^s \cap \widehat{Z}_1 = \overline{Z}_1^s \cap \overline{\pi}^{-1}(\widehat{Z}_0)$$

(iv) *The C_{m_∞} -action on \overline{Z}_1 and its good quotient $\overline{\pi}_1: \overline{Z}_1 \rightarrow \overline{Z}_1$ are given by*

$$\begin{aligned} \zeta \cdot (z_1, \dots, z_r, z_\infty) &= (z_1, \dots, z_r, \zeta z_\infty), \\ \overline{\pi}_1(z_1, \dots, z_r, z_\infty) &= (z_1, \dots, z_r, z_\infty^{m_\infty}). \end{aligned}$$

Proof. Everything is obvious except, maybe, the displayed equation of the third assertion. To verify it, take any face $\delta'_1 \preceq \delta_1$ such that the associated distinguished point $x_{\delta'_1}$ lies in \overline{Z}_1^s . Then we have $e_\infty \notin \delta'_1$ and $\{e_1, \dots, e_d\} \not\subseteq \delta'_1$. Using this, we see

$$\begin{aligned} x_{\delta'_1} \in \widehat{Z}_1 &\iff P_1(\delta'_1) \subseteq \tau_1 \in \Sigma_1 \\ &\iff P_1(\delta'_1) \subseteq \tau_0 \in \Sigma_0 \\ &\iff P_0 \circ G(\delta'_1) \subseteq \tau_0 \in \Sigma_0 \\ &\iff G(\delta'_1) \in \widehat{\Sigma}_0 \\ &\iff x_{\delta'_1} \in \overline{\pi}^{-1}(\widehat{Z}_0). \end{aligned}$$

□

Now we turn to the effect of the toric modifications $\pi: Z_1 \rightarrow Z_0$ as just studied on subvarieties $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$ with $\pi(X_1) = X_0$. A first step is to formulate conditions on the ambient modification such that it preserves neat embeddings.

Let Z be an irreducible variety, $X \subseteq Z$ a closed irreducible subvariety, and suppose that X and Z are both smooth in codimension one. Given an irreducible hypersurface $E \subseteq Z$ such that $D := X \cap E$ is an irreducible hypersurface in X , we say that *some local equation for $E \subseteq Z$ restricts to a local equation for $D \subseteq X$* , if there are an open subset $W \subseteq Z$ and an $f \in \mathcal{O}(W)$ with $E \cap W = \text{div}(f)$ and $W \cap D \neq \emptyset$ such that on $U := W \cap X$ we have $D \cap U = \text{div}(h)$ for the restriction $h := f|_U$.

Definition 5.4. Let $\pi: Z_1 \rightarrow Z_0$ be the toric morphism arising from a stellar subdivision of simplicial fans, T_1 the big torus of Z_1 , and $E = \overline{T_1 \cdot z} \subseteq Z_1$ the exceptional divisor. Let $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$ be irreducible subvarieties such that X_1 is smooth in codimension one and $\pi(X_1) = X_0$ holds. We say that $\pi: Z_1 \rightarrow Z_0$ is a *neat ambient modification* for $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$ if $D := X_1 \cap E$ is an irreducible hypersurface in X_1 intersecting $T_1 \cdot z$, some local equation for $E \subseteq Z_1$

restricts to a local equation for $D \subseteq X_1$, and $X_0 \cap \pi(E)$ is of codimension at least two in X_0 .

Note that for a neat ambient modification $\pi: Z_1 \rightarrow Z_0$ the subvarieties $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$ intersect the big torus orbits in Z_1 and Z_0 respectively. Moreover, X_0 inherits the property of being smooth in codimension one from X_1 .

Proposition 5.5. *Let $\pi: Z_1 \rightarrow Z_0$ be a neat ambient modification for irreducible subvarieties $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$, both being smooth in codimension one. Then the following statements are equivalent.*

- (i) $X_0 \subseteq Z_0$ is neatly embedded.
- (ii) $X_1 \subseteq Z_1$ is neatly embedded.

Proof. We denote by T_0 and T_1 the acting tori of the ambient toric varieties Z_0 and Z_1 respectively.

We prove the implication “(i) \Rightarrow (ii)”. In order to verify Property 2.5 (i) for $X_1 \subseteq Z_1$, let $E_1 = \overline{T_1 \cdot z_1}$ be any invariant prime divisor of Z_1 . If $E_1 = E$ holds, then the definition of an ambient modification guarantees that $D_1 = X_1 \cap E_1$ is as wanted.

For $E_1 \neq E$, note first that X_1 intersects $T_1 \cdot z_1$, because X_0 intersects the one-codimensional orbit $T_0 \cdot \pi(z_1)$. Thus, $D_1 = X_1 \cap E_1$ intersects $T_1 \cdot z_1$. Since Z_1 is \mathbb{Q} -factorial, E_1 is locally the zero set of a function. Consequently, D_1 is of pure codimension one in X_1 . Moreover,

$$D_1 \setminus E = X_1 \cap E_1 \setminus E = \pi^{-1}(X_0 \cap \pi(E_1) \setminus \pi(E))$$

is irreducible. Since $X_1 \cap E$ is irreducible and not contained in E_1 , we see that there are no components of D_1 inside E . Thus, D_1 is irreducible.

We verify Property 2.5 (ii) for $X_1 \subseteq Z_1$. Let $\kappa: X_1 \rightarrow X_0$ be the restriction of $\pi: Z_1 \rightarrow Z_0$. Then we have the canonical push forward homomorphisms

$$\pi_*: \text{Cl}(Z_1) \rightarrow \text{Cl}(Z_0), \quad \kappa_*: \text{Cl}(X_1) \rightarrow \text{Cl}(X_0),$$

sending the classes of the exceptional divisors $E \subseteq Z_1$ and $D = X_1 \cap E \subseteq X_1$ to zero. Denoting by $\iota_i: X_i \rightarrow Z_i$ the embeddings, Property 2.5 (i) allows us to work with the pullback maps $\iota_i^*: \text{Cl}(Z_i) \rightarrow \text{Cl}(X_i)$ as in Remark 2.4 and obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} \cdot [E] & \longrightarrow & \text{Cl}(Z_1) & \xrightarrow{\pi_*} & \text{Cl}(Z_0) \longrightarrow 0 \\ & & \downarrow \iota_1^* & & \downarrow \iota_1^* & & \downarrow \iota_0^* \cong \\ 0 & \longrightarrow & \mathbb{Z} \cdot [D] & \longrightarrow & \text{Cl}(X_1) & \xrightarrow{\kappa_*} & \text{Cl}(X_0) \longrightarrow 0 \end{array}$$

Note that $\mathbb{Z} \cdot [E] \subseteq \text{Cl}(Z_1)$ as well as $\mathbb{Z} \cdot [D] \subseteq \text{Cl}(X_1)$ are nontrivial and free. Since some local equation $f \in \mathcal{O}(W)$ of $E \subseteq Z_1$ restricts to a local equation $h \in \mathcal{O}(U)$ of $D \subseteq X_1$, we obtain

$$D \cap U = \text{div}(h) = \iota_1^*(\text{div}(f)) = (\iota_1^*(E)) \cap U$$

on the open set $U = W \cap X_1$. Since E and D are irreducible hypersurfaces, we can conclude $D = \iota_1^*(E)$. Thus, $\iota_1^*: \mathbb{Z}[E] \rightarrow \mathbb{Z}[D]$ is surjective, and hence an isomorphism. Then, by the Five Lemma, also $\iota_1^*: \text{Cl}(Z_1) \rightarrow \text{Cl}(X_1)$ is an isomorphism.

We turn to the implication “(ii) \Rightarrow (i)”. In order to verify Property 2.5 (i) for $X_0 \subseteq Z_0$, let $E_0 \subseteq Z_0$ be any invariant prime divisor. Then $E_0 = \pi(E_1)$ holds with a unique invariant prime divisor $E_1 \subseteq Z_1$. Moreover, we have

$$X_0 \cap E_0 \setminus \pi(E) = \pi(X_1 \cap E_1 \setminus E).$$

Since $X_1 \cap E_1 \setminus E$ is irreducible, the same holds for $X_0 \cap E_0 \setminus \pi(E)$. Thus, in order to see that $X_0 \cap E_0$ is irreducible, we only have to show that it has no irreducible components inside $X_0 \cap \pi(E)$. Since Z_0 is \mathbb{Q} -factorial, some multiple nE_0 is locally principal. Consequently, all components of $X_0 \cap E_0$ are hypersurfaces in X_0 and none of them can be contained in the small set $X_0 \cap \pi(E)$.

We establish Property 2.5 (ii) for $X_0 \subseteq Z_0$. Having already verified 2.5 (i), we may use the pullback map $\iota_0^*: \text{Cl}(Z_0) \rightarrow \text{Cl}(X_0)$. As before, let $\kappa: X_1 \rightarrow X_0$ be the restriction of $\pi: Z_1 \rightarrow Z_0$. Then we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} \cdot [E] & \longrightarrow & \text{Cl}(Z_1) & \xrightarrow{\pi_*} & \text{Cl}(Z_0) \longrightarrow 0 \\ & & \downarrow \iota_1^* \cong & & \downarrow \iota_1^* \cong & & \downarrow \iota_0^* \\ 0 & \longrightarrow & \mathbb{Z} \cdot [D] & \longrightarrow & \text{Cl}(X_1) & \xrightarrow{\kappa_*} & \text{Cl}(X_0) \longrightarrow 0 \end{array}$$

Again, the rows of this diagram are exact sequences. Using, e.g., the Five Lemma, we see that $\iota_0^*: \text{Cl}(Z_0) \rightarrow \text{Cl}(X_0)$ is an isomorphism. \square

Corollary 5.6. *Let Z_1, Z_0 be \mathbb{Q} -factorial toric varieties with $\Gamma(Z_i, \mathcal{O}^*) = \mathbb{K}^*$, and let $\pi: Z_1 \rightarrow Z_0$ be a neat ambient modification for $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$. Denote by $p_i: \widehat{Z}_i \rightarrow Z_i$ the toric Cox constructions and set $\widehat{X}_i := p_i^{-1}(X_i)$.*

- (i) *If $X_0 \subseteq Z_0$ is neatly embedded, X_1 is normal with $\Gamma(X_1, \mathcal{O}^*) = \mathbb{K}^*$ and Z_1 is smooth, then $p_1: \widehat{X}_1 \rightarrow X_1$ is a universal torsor of X_1 .*
- (ii) *If $X_1 \subseteq Z_1$ is neatly embedded X_0 is normal with $\Gamma(X_0, \mathcal{O}^*) = \mathbb{K}^*$ and Z_0 is smooth, then $p_0: \widehat{X}_0 \rightarrow X_0$ is a universal torsor of X_0 .*

Proof. The statements are a direct consequence of Proposition 5.5 and Corollary 2.8. \square

We investigate the effect of a neat ambient modification on the total coordinate space and the Cox ring. Let $\pi: Z_1 \rightarrow Z_0$ be the toric morphism associated to a stellar subdivision of simplicial fans. Then Proposition 5.2 provides a commutative diagram

$$\begin{array}{ccccc} & & \overline{Z}_1 & & \\ & \swarrow \overline{\pi}_1 & \uparrow & \searrow \overline{\pi} & \\ \overline{Z}_1 & & \widehat{Z}_1 & & \overline{Z}_0 \\ & \swarrow \widehat{\pi}_1 & & \searrow \widehat{\pi} & \\ \widehat{Z}_1 & & & & \widehat{Z}_0 \\ \uparrow p_1 & & & & \uparrow p_0 \\ Z_1 & \xrightarrow{\pi} & & & Z_0 \end{array}$$

where $p_i: \widehat{Z}_i \rightarrow Z_i$ are universal torsors, $\overline{\pi}: \overline{Z}_1 \rightarrow \overline{Z}_0$ is the good quotient for a \mathbb{K}^* -action on \overline{Z}_1 , and $\overline{\pi}_1: \overline{Z}_1 \rightarrow \overline{Z}_1$ is the quotient for the action of the group C_{m_∞} of m_∞ -th roots of unity, where m_∞ denotes the index of the stellar subdivision.

Now, suppose that $\pi: Z_1 \rightarrow Z_0$ is a neat ambient modification for two irreducible subvarieties $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$. By definition, we have $X_0 = \pi(X_1)$ and both, X_1 and X_0 , are smooth in codimension one. Consider the restriction $\kappa: X_1 \rightarrow X_0$ of $\pi: Z_1 \rightarrow Z_0$, the inverse images $\widehat{X}_i := p_i^{-1}(X_i)$, their closures $\overline{X}_i \subseteq \overline{Z}_i$, the inverse image $\widehat{Y}_1 := \widehat{\pi}_1^{-1}(\widehat{X}_1)$, its closure $\overline{Y}_1 \subseteq \overline{Z}_1$ and the restrictions κ , $\widehat{\kappa}$ and $\overline{\kappa}$ of π , $\widehat{\pi}$ and $\overline{\pi}$ as well as $\widehat{\kappa}_1$ and $\overline{\kappa}_1$ of $\widehat{\pi}_1$ and $\overline{\pi}_1$ respectively.

Lemma 5.7. *In the above setting, $\overline{Y}_1 \subseteq \overline{Z}_1$ is C_{m_∞} -invariant and \mathbb{K}^* -invariant. If $\overline{T}_0 \subseteq \overline{Z}_0$ denotes the big torus orbit, then one has*

$$\overline{Y}_1 = \pi_1^{-1}(\overline{X}_1) = \overline{\pi^{-1}(\overline{X}_0 \cap \overline{T}_0)}.$$

If one of the embeddings $X_i \subseteq Z_i$ is neat, then \overline{Y}_1 is irreducible, and one has a commutative diagram

$$\begin{array}{ccccc}
 & & \overline{Y}_1 & & \\
 & \swarrow /C_{m_\infty} & \uparrow & \searrow //\mathbb{K}^* & \\
 \overline{X}_1 & & \widehat{Y}_1 & & \overline{X}_0 \\
 & \swarrow \widehat{\kappa}_1 & & \searrow \widehat{\kappa} & \\
 \widehat{X}_1 & & & & \widehat{X}_0 \\
 \uparrow & & & & \uparrow \\
 \widehat{X}_1 & & & & \widehat{X}_0 \\
 \downarrow p_1 & & & & \downarrow p_0 \\
 X_1 & \xrightarrow{\kappa} & & & X_0
 \end{array}$$

Proof. Let $T_i \subseteq Z_i$ denote the respective big torus orbits. The first statement follows from the fact that $\overline{Y}_1 \subseteq \overline{Z}$ is the closure of the following C_{m_∞} -invariant and \mathbb{K}^* -invariant subset

$$\pi_1^{-1}(p_1^{-1}(\pi^{-1}(X_0 \cap T_0))) = \pi^{-1}(p_0^{-1}(X_0 \cap T_0)) \subseteq \overline{Y}_1.$$

If one of the embeddings $X_i \subseteq Z_i$ is neat, then Proposition 5.5 ensures that $X_0 \subseteq Z_0$ is neat. Theorem 2.6 then guarantees that $p_0^{-1}(X_0 \cap T_0)$ is irreducible. This gives the second claim. The commutative diagram is then obvious. \square

This statement shows in particular that \overline{Y}_1 can be explicitly computed from either \overline{X}_1 , or \overline{X}_0 and vice versa. Cutting down the minus cell \overline{Z}_1^- of the \mathbb{K}^* -action on \overline{Z}_1 , we obtain the minus cell for the induced \mathbb{K}^* -action on \overline{Y}_1 :

$$\overline{Y}_1^- = \overline{Y}_1 \cap \overline{Z}_1^-.$$

Since $\pi_1^{-1}(\overline{Z}_1^-) = \overline{Z}_1^-$ holds and π_1^{-1} equals the identical map along \overline{Z}_1^- , we see that the group H_1 , acting on \overline{Z}_1 , leaves \overline{Y}_1^- invariant.

Definition 5.8. We say that the neat ambient modification $\pi: Z_1 \rightarrow Z_0$ for $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$ is *controlled* if the minus cell \overline{Y}_1^- is H_1 -irreducible, i.e., H_1 acts transitively on the collection of irreducible components of \overline{Y}_1^- .

Note that in the case of a torus H_1 , being controlled just means that the minus cell \overline{Y}_1^- is irreducible. For our first main result on modifications, we recall the following notation: given a toric variety Z with toric prime divisors $E_1, \dots, E_r \subseteq Z_0$ and Cox construction $p: \widehat{Z} \rightarrow Z$ and a closed subvariety $X \subseteq Z$, we set

$$W_Z := Z \setminus \bigcup_{i \neq j} E_i \cap E_j, \quad W_{\overline{Z}} := p^{-1}(W_Z),$$

$$W_X := W_Z \cap \widehat{X} \quad W_{\overline{X}} := W_{\overline{Z}} \cap \widehat{X}.$$

Theorem 5.9. *Let Z_1, Z_0 be \mathbb{Q} -factorial toric varieties with $\Gamma(Z_i, \mathcal{O}^*) = \mathbb{K}^*$, and let $\pi: Z_1 \rightarrow Z_0$ be a neat ambient modification for $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$, where $X_0 \subseteq Z_0$ is neatly embedded and W_{X_1} is normal with $\Gamma(W_{X_1}, \mathcal{O}^*) = \mathbb{K}^*$. Suppose that $\overline{X}_0 \setminus \widehat{X}_0$ is of codimension at least two in \overline{X}_0 and that $\pi: Z_1 \rightarrow Z_0$ is controlled.*

- (i) *The complement $\overline{X}_1 \setminus \widehat{X}_1$ is of codimension at least two in \overline{X}_1 . Moreover, if \widehat{X}_1 is normal, then so is X_1 and $\widehat{X}_1 \rightarrow X_1$ is a universal torsor of X_1 .*

- (ii) *The normalization of X_1 has finitely generated Cox ring and it has the H_1 -equivariant normalization of \overline{X}_1 as its total coordinate space.*

Note that if the embedding $X_0 \subseteq Z_0$ is obtained as in Construction 3.13, then any neat controlled ambient modification $\pi: Z_1 \rightarrow Z_0$ for normal subvarieties $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$ fullfills the assumption of Theorem 5.9. Before giving the proof, we note an immediate consequence.

Corollary 5.10. *Let Z_1, Z_0 be \mathbb{Q} -factorial toric varieties with $\Gamma(Z_i, \mathcal{O}^*) = \mathbb{K}^*$ and $\pi: Z_1 \rightarrow Z_0$ a neat controlled ambient modification for $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$, where $X_0 \subseteq Z_0$ is normal, neatly embedded and $\overline{X}_0 \setminus \widehat{X}_0$ is of codimension at least two in \overline{X}_0 . If \overline{Y}_1 is normal, then X_1 is normal with universal torsor $\widehat{X}_1 \rightarrow X_1$ and it has \overline{X}_1 together with the induced H_1 -action as its total coordinate space.*

Proof. The fact that \overline{Y}_1 is normal implies that $\overline{X}_1 = \overline{Y}_1/C_{m_\infty}$ and $X_1 = \widehat{X}_1/H_1$ are normal. Thus, we may apply Theorem 5.9. \square

The proof of Theorem 5.9 uses the following observation on \mathbb{K}^* -varieties Y ; recall, that we denote the fixed point set by Y^0 and the set of closed one-dimensional orbits by Y^s .

Lemma 5.11. *Let Y be an irreducible affine \mathbb{K}^* -variety with quotient $\pi: Y \rightarrow Y//\mathbb{K}^*$ such that $\pi(Y^0)$ is small in $Y//\mathbb{K}^*$ and $Y \setminus Y^-$ is affine. Then, for any open invariant $V \subseteq Y$ with $Y^s \setminus V$ small in Y , the following statements are equivalent.*

- (i) *The complement $Y \setminus V$ is small in Y .*
- (ii) *The complement $Y \setminus V$ comprises no component of Y^- .*

Proof. Suppose that (i) holds. Since $Y \setminus Y^-$ is affine, every component of the minus cell Y^- is a hypersurface. In particular, the small complement $Y \setminus V$ cannot contain any component of Y^- .

Now suppose that (ii) holds. The complement of V in Y can be decomposed as follows into locally closed subsets:

$$Y \setminus V = Y^s \setminus V \cup Y^- \setminus V \cup Y^+ \setminus V.$$

By assumption, the first two sets are small in Y . So, it suffices to show that Y^+ is small in Y . Since $Y \setminus Y^-$ is affine, the same holds for $(Y \setminus Y^-)/\mathbb{K}^*$. Consequently, in the commutative diagram

$$\begin{array}{ccc} Y \setminus Y^- & \subseteq & Y \\ \downarrow \pi_- / \mathbb{K}^* & & \downarrow \pi / \mathbb{K}^* \\ (Y \setminus Y^-)/\mathbb{K}^* & \longrightarrow & Y//\mathbb{K}^* \end{array}$$

the induced (projective) morphism of quotient spaces is finite. Thus, since $\pi(Y^0)$ is small in $Y//\mathbb{K}^*$, we see that $\pi_-(Y^+)$ is small in $(Y \setminus Y^-)/\mathbb{K}^*$. Consequently, $Y^+ \setminus Y^-$ must be small in $Y \setminus Y^-$. This gives the assertion. \square

Proof of Theorem 5.9. Only the assertion that $\overline{X}_1 \setminus \widehat{X}_1$ is small, needs a proof; the remaining statements are direct applications of Corollary 2.7. In order to verify that $\overline{X}_1 \setminus \widehat{X}_1$ is small, it suffices to show that $\overline{Y}_1 \setminus \widehat{Y}_1$ is small. The idea is to apply Lemma 5.11 to the \mathbb{K}^* -variety \overline{Y}_1 . Let us check the assumptions.

Since $X_0 \subseteq Z_0$ is neatly embedded and p_0 is equidimensional, the complement $\widehat{X}_0 \setminus W_{\overline{X}_0} = p_0^{-1}(X_0 \setminus W_{X_0})$ is small in \widehat{X}_0 . Denote by $D \subseteq X_1$ the exceptional divisor. Then Lemma 5.3 (iii) tells us that the image of the fixed point set under the quotient map $\overline{\kappa}: \overline{Y}_1 \rightarrow \overline{X}_0$ satisfies

$$\overline{\kappa}(\overline{Y}_1^0) \cap \widehat{X}_0 \subseteq p_0^{-1}(\pi(D)) \subseteq \widehat{X}_0 \setminus W_{\overline{X}_0}.$$

As observed before, the latter set is small, and, moreover $\overline{X}_0 \setminus \widehat{X}_0$ is small by assumption. Consequently $\overline{\kappa}(\overline{Y}_1^0)$ is small in \overline{X}_0 . Since, in the notation of Lemma 5.3, we have $\overline{Y}_1^- = V(\overline{Y}_1, z_\infty)$, the set $\overline{Y}_1 \setminus \overline{Y}_1^-$ is affine. Moreover, again by Lemma 5.3 (iii), we have

$$\overline{Y}_1^s \cap \widehat{Y}_1 = \overline{Y}_1 \cap \overline{Z}_1^s \cap \widehat{Z}_1 = \overline{Y}_1 \cap \overline{Z}_1^s \cap \overline{\pi}^{-1}(\widehat{Z}_0) = \overline{Y}_1^s \cap \overline{\kappa}^{-1}(\widehat{X}_0).$$

Since $\kappa: \overline{Y}_1^s \rightarrow \kappa(\overline{Y}_1^s)$ has constant fibre dimension (one) and $\overline{X}_0 \setminus \widehat{X}_0$ is small, we can conclude that $\overline{Y}_1^s \setminus \widehat{Y}_1$ is small. Since \overline{Y}_1^- intersects \widehat{Y}_1 , and the ambient modification $Z_1 \rightarrow Z_0$ is controlled, we obtain that every component of \overline{Y}_1^- intersects \widehat{Y}_1 . Thus, we may apply Lemma 5.11, and obtain that $\overline{Y}_1 \setminus \widehat{Y}_1$ is small. \square

Our second main result on modifications treats the case of contractions, i.e., we start with a neatly embedded subvariety $X_1 \subseteq Z_1$.

Theorem 5.12. *Let $\pi: Z_1 \rightarrow Z_0$ the toric morphism associated to a stellar subdivision of simplicial fans and suppose $\Gamma(Z_i, \mathcal{O}^*) = \mathbb{K}^*$. Let $X_1 \subseteq Z_1$ be neatly embedded with W_{X_1} normal and $\Gamma(W_{X_1}, \mathcal{O}^*) = \mathbb{K}^*$. Set $X_0 := \pi(X_1)$ and suppose that $\overline{X}_1 \setminus \widehat{X}_1$ is of codimension at least two in \overline{X}_1 .*

- (i) *The embedding $X_0 \subseteq Z_0$ is neat and $\pi: Z_1 \rightarrow Z_0$ is a neat controlled ambient modification for $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$.*
- (ii) *The complement $\overline{X}_0 \setminus \widehat{X}_0$ is of codimension at least two in \overline{X}_0 . If \widehat{X}_0 is normal, then so is X_0 and $\widehat{X}_0 \rightarrow X_0$ is a universal torsor of X_0 .*
- (iii) *The normalization of X_0 has finitely generated Cox ring and it has the H_0 -equivariant normalization of \overline{X}_0 as its total coordinate space.*

Again we state a consequence before proving the result; note that, below, if the neat embedding $X_1 \subseteq Z_1$ is obtained by Construction 3.13 and the index of the ambient modification equals one, then all assumptions are satisfied.

Corollary 5.13. *Let Z_1, Z_0 be \mathbb{Q} -factorial toric varieties with $\Gamma(Z_i, \mathcal{O}^*) = \mathbb{K}^*$, and let $\pi: Z_1 \rightarrow Z_0$ be a neat ambient modification for $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$, where $X_1 \subseteq Z_1$ is normal, neatly embedded and $\overline{X}_1 \setminus \widehat{X}_1$ is of codimension at least two in \overline{X}_1 . Then, if \overline{Y}_1 is normal, X_0 is normal with universal torsor $\widehat{X}_0 \rightarrow X_0$ and \overline{X}_0 together with the induced H_0 -action as its total coordinate space.*

Proof. The fact that \overline{Y}_1 is normal implies that $\overline{X}_0 = \overline{Y}_1/\mathbb{K}^*$ and $X_0 = \widehat{X}_0/H_0$ are normal. Thus, we may apply Theorem 5.12. \square

Similarly as for our first main result, we need also for proving the second one an observation on \mathbb{K}^* -actions.

Lemma 5.14. *Let Y be an irreducible affine \mathbb{K}^* -variety admitting nontrivial homogeneous global regular functions of positive degree. Then, for any irreducible component $B \subseteq Y^0$, we have $B \subseteq \overline{Y^+} \setminus Y^0$.*

Proof. Since there are functions of positive degree, we have $Y \neq Y^-$. Thus, we have the following commutative diagram with the induced projective morphism φ of quotient spaces

$$\begin{array}{ccc} Y \setminus Y^- & \subseteq & Y \\ \downarrow \scriptstyle / \mathbb{K}^* \pi_- & & \downarrow \scriptstyle \pi // \mathbb{K}^* \\ (Y \setminus Y^-) / \mathbb{K}^* & \xrightarrow{\varphi} & Y // \mathbb{K}^* \end{array}$$

Now, π maps Y^0 bijectively onto $\pi(Y^0)$. Hence $\pi(B)$ is an irreducible component of $\pi(Y^0)$. Thus, there is an irreducible component $A \subseteq \pi^{-1}(\varphi^{-1}(\pi(B)))$ with $\pi(A) = \pi(B)$. For this component, we have $A \subseteq Y^+ \setminus Y^0$ and $B \subseteq \overline{A}$. \square

Proof of Theorem 5.12. Let $E \subseteq Z_1$ be the exceptional divisor. Since $X_1 \subseteq Z_1$ is neatly embedded, $D := X_1 \cap E$ is an irreducible hypersurface in X_1 . Its inverse image with respect to $p_1: \hat{X}_1 \rightarrow X_1$ is the irreducible hypersurface

$$p_1^{-1}(D) = \hat{X}_1 \cap \overline{Z}_1^- = \pi_1(\hat{Y}_1 \cap \overline{Y}_1^-) \subseteq \hat{X}_1.$$

Since π_1 equals the identity along $\pi_1^{-1}(\overline{Z}_1^-) = \overline{Z}_1^-$, the set $\hat{Y}_1 \cap \overline{Y}_1^-$ is H_1 -irreducible. By Lemma 5.3 (ii), the minus cell \overline{Y}_1^- is the zero set of a function and hence of pure codimension one. Since $\overline{X}_1 \setminus \hat{X}_1$ is small, the same holds for $\overline{Y}_1 \setminus \hat{Y}_1$, and we can conclude that also \overline{Y}_1^- is H_1 -irreducible.

In order to obtain that $\pi: Z_1 \rightarrow Z_0$ is a neat in controlled ambient modification, we still have to show that $D = X_1 \cap E$ has a small image $\pi(D)$ in X_0 . Using Lemma 5.3 (ii) we observe

$$\pi(D) = \pi(X_1 \cap E) = \pi\left(p_1\left(\pi_1\left(\hat{Y}_1 \cap \overline{Y}_1^-\right)\right)\right) \subseteq p_0\left(\pi\left(\overline{Y}_1^-\right) \cap \hat{X}_0\right).$$

Thus, we have to show that $\pi(\overline{Y}_1^-) = \pi(\overline{Y}_1^0)$ is of codimension at least two in \overline{X}_0 . For this, it suffices to show that \overline{Y}_1^0 is of codimension at least three in \overline{X}_0 . But this follows from the facts that, according to Lemma 5.14, any component of \overline{Y}_1^0 is a proper subset of a component of \overline{Y}_1^+ , and that we have $\overline{Y}_1^+ \subseteq \overline{Y}_1 \setminus \hat{Y}_1$, where the latter set is small in \overline{Y}_1 .

Let us see, why $\overline{X}_0 \setminus \hat{X}_0$ is small in \overline{X}_0 . First note that we have $\pi(\hat{Y}_1) \subseteq \hat{X}_0$. Thus, if $\overline{X}_0 \setminus \hat{X}_0$ would contain a divisor, then also $\overline{Y}_1 \setminus \hat{Y}_1$ and hence $\overline{X}_1 \setminus \hat{X}_1$ must contain a divisor, a contradiction. All remaining statements follow from Corollary 2.7. \square

6. COMBINATORIAL CONTRACTION

We take a closer look at \mathbb{Q} -factorial projective varieties with finitely generated Cox ring. Corollaries 4.3 and 4.10 tell us that all those sharing a common Cox ring have the same moving cone and their semiample cones are the full dimensional cones of a fan subdivision of this common moving cone. Moreover, any two of them differ only by a small birational transformation, i.e., a birational map, which is an isomorphism in codimension one.

Conversely, if any two \mathbb{Q} -factorial projective varieties with finitely generated Cox ring differ only by a small birational transformation then their Cox rings are isomorphic, and hence, up to isomorphism, both of them show up in a common picture as above. In this section, we link all these pictures to a larger one by studying certain divisorial contractions. The following concepts will be crucial.

Definition 6.1. Let X be \mathbb{Q} -factorial projective variety with finitely generated Cox ring.

- (i) We call a class $[D] \in \text{Cl}(X)$ *combinatorially contractible* if it generates an extremal ray of the effective cone of X and, for some representative D , all vector spaces $\Gamma(X, \mathcal{O}(nD))$, where $n > 0$, are of dimension one.
- (ii) We say that the variety X is *combinatorially minimal* if it has no combinatorially contractible divisor classes.

As we will see, combinatorial minimality of X is equivalent to saying that the moving cone of X equals its effective cone. Moreover, a surface X turns out to be combinatorially minimal if and only if its semiample cone equals its effective cone; for example, $\mathbb{P}_1 \times \mathbb{P}_1$ is combinatorially minimal, whereas the first Hirzebruch surface is not. The following theorem shows that one may reduce any X in a very controlled process to a combinatorially minimal one.

Theorem 6.2. *For every \mathbb{Q} -factorial projective variety X with finitely generated Cox ring arises from a combinatorially minimal one X_0 via a finite sequence*

$$X = X'_n - - \triangleright X_n \longrightarrow X'_{n-1} - - \triangleright X_{n-1} \longrightarrow \cdots \longrightarrow X'_0 = X_0$$

where $X'_i - - \triangleright X_i$ is a small birational transformation and $X_i \longrightarrow X'_{i-1}$ comes from a neat controlled ambient modification of \mathbb{Q} -factorial projective toric varieties.

The crucial part of the proof of this statement is an explicit contraction criterion for varieties arising from bunched rings, see Proposition 6.7. There, we also explicitly describe the contracting ambient modification, which in turn allows computation of the Cox rings in the contraction steps.

Let (R, \mathfrak{F}, Φ) be a bunched ring with $\mathfrak{F} = \{f_1, \dots, f_r\}$ and $(E \xrightarrow{Q} K, \gamma)$ as its projected cone. The diagonalizable group $H := \text{Spec}(\mathbb{K}[K])$ acts on $\overline{X} := \text{Spec}(R)$. Defining a K -grading on $\mathbb{K}[T_1, \dots, T_r]$ by $\deg(T_i) := w_i =: \deg(f_i)$, we obtain a graded epimorphism

$$\mathbb{K}[T_1, \dots, T_r] \rightarrow R, \quad T_i \mapsto f_i.$$

This gives rise to an H -equivariant closed embedding $\overline{X} \rightarrow \overline{Z}$, where $\overline{Z} = \mathbb{K}^r$. Note that the actions of H on \overline{X} and \overline{Z} define two collections of orbit cones, namely $\Omega_H(\overline{X})$ and $\Omega_H(\overline{Z})$. These collections in turn give rise to two GIT-fans, $\Lambda(\overline{X})$ and $\Lambda(\overline{Z})$. Since $\Omega_H(\overline{X}) \subseteq \Omega_H(\overline{Z})$ holds, $\Lambda(\overline{Z})$ refines $\Lambda(\overline{X})$.

Let us briefly recall how to construct neat embeddings in terms of $\Lambda(\overline{Z})$. The map $Q: E \rightarrow K$, $e_i \mapsto w_i$ defines two mutually dual exact sequences of vector spaces

$$0 \longrightarrow L_{\mathbb{Q}} \longrightarrow F_{\mathbb{Q}} \xrightarrow{P} N_{\mathbb{Q}} \longrightarrow 0$$

$$0 \longleftarrow K_{\mathbb{Q}}^0 \xleftarrow{Q} E_{\mathbb{Q}} \longleftarrow M_{\mathbb{Q}} \longleftarrow 0$$

As earlier, write $e_1, \dots, e_r \in F$ for the dual basis, set $\delta = \gamma^\vee$ and $v_i := P(e_i) \in N$. Note that dealing with projective X and Z amounts to requiring that the image $Q^0(\gamma) \subseteq K_{\mathbb{Q}}^0$ is pointed. There is an injection

$$\begin{aligned} \Lambda(\overline{Z}) &\longrightarrow \{\text{maximal compatible } \widehat{\mathfrak{S}} \subseteq \text{faces}(\delta)\} \\ \eta &\mapsto \{\gamma_0^*; \lambda^\circ \subseteq Q^0(\gamma_0)^\circ\} \end{aligned}$$

and the maximal projectable collections $\widehat{\mathfrak{S}}$ assigned to the $\eta \in \Lambda(\overline{Z})$ are precisely those having a polytopal image fan Σ in N with its rays among $\varrho_i := \mathbb{Q}_{\geq 0} \cdot v_i$, where $1 \leq i \leq r$. In fact, Σ is the normal fan of the fiber polytope $(Q^0)^{-1}(w^0) \cap \gamma$ for any $w^0 \in \eta^\circ$. As a consequence of Construction 3.13 and Proposition 3.14, we note the following.

Remark 6.3. Suppose that the variety X associated to (R, \mathfrak{F}, Φ) is \mathbb{Q} -factorial and projective. Then, for any $\eta \in \Lambda(\overline{Z})$ with $\eta^\circ \subseteq \text{Ample}(X)$, we obtain a neat embedding $X \rightarrow Z$ into the toric variety Z arising from the image fan of η . Moreover, Z is \mathbb{Q} -factorial and projective. We call the neat embeddings $X \rightarrow Z$ constructed this way *associated neat embeddings*.

We are ready to introduce the “bunch theoretical” formulation of combinatorial contractibility. In the above setup, it is the following.

Definition 6.4. We say that a weight $w_i = \deg(f_i) \in K$ is *exceptional*, if $\mathbb{Q}_{\geq 0} \cdot w_i$ is an extremal ray of $Q^0(\gamma)$ and for any $j \neq i$ we have $w_j \notin \mathbb{Q}_{> 0} \cdot w_i$.

Example 6.5. Consider the two gradings of the polynomial ring $\mathbb{K}[T_1, \dots, T_4]$ assigning T_i the degree $w_i \in \mathbb{Z}^2$ as follows; the shaded area indicates $Q^0(\gamma)$.



Then in the first one w_1 is not exceptional, but in the second one it is. Note that these configurations define $\mathbb{P}_1 \times \mathbb{P}_1$ and the first Hirzebruch surface.

Lemma 6.6. Consider a chamber $\eta_1 \in \Lambda(\overline{Z})$, the corresponding maximal projectable fan $\widehat{\Sigma}_1$ and the associated image fan Σ_1 . Suppose that

$$\dim(\eta_1) = \dim(K_{\mathbb{Q}}), \quad \eta_1^\circ \subseteq \bigcap_{j=1}^r \text{cone}(e_l; l \neq j)^\circ$$

holds, i.e., the fan Σ_1 is simplicial and all $\varrho_1, \dots, \varrho_r$ belong to Σ_1 , and fix any index i with $1 \leq i \leq r$. Then the following statements are equivalent.

- (i) There is a polytopal simplicial fan Σ_0 such that Σ_1 is the stellar subdivision of Σ_0 at the vector v_i .
- (ii) The weight w_i is exceptional, there is a chamber $\eta_0 \in \Lambda(\overline{Z})$ of full dimension, with $w_i^0 \in \eta_0$ and $\eta_0 \cap \eta_1$ is a facet of both, η_0 and η_1 .

If one of these statements holds, then one may choose Σ_0 of (i) to be the image fan of the maximal projectable fan associated to η_0 of (ii), and, moreover, one has

$$\eta_0 \cap \eta_1 = \eta_0 \cap \text{cone}(w_1^0, \dots, w_{i-1}^0, w_{i+1}^0, \dots, w_r^0).$$

Proof. For the sake of a simple notation, we rename the index i in question to ∞ and renumber our vectors in N to $v_1, \dots, v_s, v_\infty$.

Suppose that (i) holds. Then we obtain the following maximal projectable subfans of the fan of faces of the positive orthant $\delta = \text{cone}(e_1, \dots, e_s, e_\infty)$ in $F_{\mathbb{Q}}$:

$$\begin{aligned} \widehat{\Sigma}_0 &:= \{\delta_0 \preceq \text{cone}(e_1, \dots, e_s); P(\delta_0) \in \Sigma_0\}, \\ \widehat{\Sigma}_1 &:= \{\delta_0 \preceq \text{cone}(e_1, \dots, e_s, e_\infty); P(\delta_0) \in \Sigma_1\}, \\ \widehat{\Sigma}_{01} &:= \{\delta_0 \preceq \text{cone}(e_1, \dots, e_s, e_\infty); P(\delta_0) \subseteq \sigma \in \Sigma_0\}. \end{aligned}$$

Note that $\widehat{\Sigma}_0$ and $\widehat{\Sigma}_{01}$ have Σ_0 as their image fan, whereas $\widehat{\Sigma}_1$ has Σ_1 as its image fan. Let $\widehat{\Theta}_0, \widehat{\Theta}_1$ and $\widehat{\Theta}_{01}$ denote the compatible collections of faces of $\gamma \subseteq E_{\mathbb{Q}}$ corresponding to $\widehat{\Sigma}_0, \widehat{\Sigma}_1$ and $\widehat{\Sigma}_{01}$. The associated chambers in $\Lambda(\overline{Z})$ are given by

$$\eta_0 = \bigcap_{\widehat{\theta}_0 \in \widehat{\Theta}_0} Q^0(\widehat{\theta}_0), \quad \eta_1 = \bigcap_{\widehat{\theta}_1 \in \widehat{\Theta}_1} Q^0(\widehat{\theta}_1), \quad \eta_{01} = \bigcap_{\widehat{\theta}_{01} \in \widehat{\Theta}_{01}} Q^0(\widehat{\theta}_{01}).$$

Moreover, as $\widehat{\Sigma}_0$ and $\widehat{\Sigma}_1$ are subfans of $\widehat{\Sigma}_{01}$, Proposition 3.9 tells us that η_{01} is a face of both, η_0 and η_1 .

Since the fan Σ_0 is simplicial, the map F is injective along the cones of $\widehat{\Sigma}_0$, which in turn implies that η_0 is of full dimension. Moreover, since $\mathbb{Q}_{\geq 0} \cdot e_\infty \notin \widehat{\Sigma}_0$ holds, we obtain

$$w_\infty^0 \in \eta_0, \quad \eta_0 \not\subseteq \text{cone}(w_1^0, \dots, w_s^0).$$

In particular, we see that w_∞^0 is exceptional. The only thing, which remains to be verified is that η_{01} is of codimension one in $K_{\mathbb{Q}}^0$. For this, let $\sigma \in \Sigma_0$ be any cone with $\varrho_\infty \subseteq \sigma$. Then σ is generated by some v_{i_1}, \dots, v_{i_d} , where $1 \leq i_d \leq r$. We set

$$\widehat{\sigma}_0 := \text{cone}(e_{i_1}, \dots, e_{i_d}) \in \widehat{\Sigma}_0, \quad \widehat{\sigma}_{01} := \text{cone}(e_{i_1}, \dots, e_{i_d}, e_\infty) \in \widehat{\Sigma}_{01},$$

Using this notation, we obtain the following decompositions of the chamber η_{01} and its relative interior:

$$\begin{aligned}\eta_{01} &= \bigcap_{\varrho_\infty \not\subseteq \sigma \in \Sigma_0^{\max}} Q^0(\hat{\sigma}_0^*) \cap \bigcap_{\varrho_\infty \subseteq \sigma \in \Sigma_0^{\max}} Q^0(\hat{\sigma}_{01}^*), \\ \eta_{01}^\circ &= \bigcap_{\varrho_\infty \not\subseteq \sigma \in \Sigma_0^{\max}} Q^0(\hat{\sigma}_0^*)^\circ \cap \bigcap_{\varrho_\infty \subseteq \sigma \in \Sigma_0^{\max}} Q^0(\hat{\sigma}_{01}^*)^\circ.\end{aligned}$$

For any $\sigma \in \Sigma_0$, the cone $Q^0(\hat{\sigma}_0^*)$ is of full dimension in $K_{\mathbb{Q}}^0$. For a cone $\sigma \in \Sigma_0$ with $\varrho_\infty \subseteq \sigma$, the cone $Q^0(\hat{\sigma}_{01}^*)$ must be of codimension one in $K_{\mathbb{Q}}^0$, because it is not of full dimension and generated by one ray less than $Q^0(\hat{\sigma}_0^*)$.

Now, there is a unique minimal $\kappa \in \Sigma_0$ comprising ϱ_∞ . For any further $\sigma \in \Sigma_0$ comprising ϱ_∞ , we have $Q^0(\hat{\sigma}_{01}^*) \subseteq Q^0(\hat{\kappa}_{01}^*)$. Thus, the above decompositions of η_{01} and η_{01}° show that η_{01} is of codimension one.

Suppose that (ii) holds. Let $\hat{\Sigma}_0$ denote the maximal projectable fan corresponding to η_0 , and let Σ_0 be its image fan in N . Note that Σ_0 is simplicial, since η_0 is of full dimension.

Since w_∞^0 is exceptional and belongs to η_0 , we have $\eta_0 \not\subseteq \text{cone}(w_1^0, \dots, w_s^0)$. Moreover, $\mathbb{Q}_{\geq 0} \cdot e_\infty \in \hat{\Sigma}_1$ implies $\eta_1 \subseteq \text{cone}(w_1^0, \dots, w_s^0)$. Thus, we obtain

$$\eta_0 \cap \eta_1 \subseteq \eta_0 \cap \text{cone}(w_1^0, \dots, w_s^0).$$

By dimension reasons, this must even be an equality. Moreover, we note that the hyperplane H_{01} in $K_{\mathbb{Q}}^0$ separating η_0 and η_1 does not contain w_∞^0 .

We show now that any ϱ_i with $1 \leq i \leq s$ must belong to Σ_0 . Suppose that some ϱ_i does not. As before, we obtain

$$\eta_0 \cap \eta_1 = \eta_0 \cap \text{cone}(w_1^0, \dots, w_{i-1}^0, w_{i+1}^0, \dots, w_s^0, w_\infty^0).$$

The left hand side is contained in the separating hyperplane H_{01} , and the right hand side contains w_∞^0 . A contradiction.

Now consider the stellar subdivision Σ_2 of Σ_0 in ϱ_∞ . Then Σ_2 is a polytopal simplicial fan. Thus

$$\hat{\Sigma}_2 := \{\delta_0 \preceq \text{cone}(e_1, \dots, e_s, e_\infty); P(\delta_0) \in \Sigma_2\},$$

arises from a full dimensional chamber $\eta_2 \in \Lambda(\bar{\Sigma})$. As seen in the proof of “(i) \Rightarrow (ii)”, the chambers η_0 and η_2 share a common facet. This gives

$$\eta_0 \cap \eta_2 = \eta_0 \cap \text{cone}(w_1^0, \dots, w_s^0) = \eta_0 \cap \eta_1.$$

Since the cones η_0 , η_1 and η_2 all show up in a common fan, we may conclude $\eta_1 = \eta_2$. This implies $\Sigma_1 = \Sigma_2$. \square

Proposition 6.7. *Let X_1 be a \mathbb{Q} -factorial projective variety arising from a bunched ring (R, \mathfrak{F}, Φ) , and let $D_1^i \subseteq X_1$ be the prime divisor given by $p_{X_1}^* D_1^i = \text{div}(f_i)$ for the generator $f_i \in \mathfrak{F}$. Then the following statements are equivalent.*

- (i) *There are associated neat embeddings $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$ into \mathbb{Q} -factorial projective toric varieties and a neat controlled ambient modification $Z_1 \rightarrow Z_0$ for $X_1 \subseteq Z_1$ and $X_0 \subseteq Z_0$ contracting D_1^i .*
- (ii) *The weight $w_i := \deg(f_i) \in K$ is exceptional and $w_i^0 \in \lambda_0$ holds for a full dimensional $\lambda_0 \in \Lambda(\bar{X})$ having a common facet with $\lambda_1 := \text{Sample}(X_1)$.*

If (ii) holds, then any pair $\eta_0, \eta_1 \in \Lambda(\bar{Z}_1)$ of adjacent fulldimensional chambers with $\eta_i^\circ \subseteq \lambda_i^\circ$ defines a neat controlled ambient modification $Z_1 \rightarrow Z_0$ as in (i). Moreover, X_0 is a \mathbb{Q} -factorial projective variety with finitely generated Cox ring.

Proof. Suppose that (i) holds. Then the ambient modification $Z_1 \rightarrow Z_0$ comes from a stellar subdivision $\Sigma_1 \rightarrow \Sigma_0$ at $v_i := P(e_i)$. According to Lemma 6.6, the stellar subdivision $\Sigma_0 \rightarrow \Sigma_1$ corresponds to a pair $\eta_0, \eta_1 \in \Lambda(\overline{Z}_1)$ of full dimensional chambers having a common facet and $w_i \in \eta_0$. Then we have $\eta_i \subseteq \lambda_i$ with unique $\lambda_i \in \Lambda(\overline{X}_1)$, and these λ_i are obviously as wanted.

Suppose that (ii) holds. Choose any full dimensional $\eta_0 \in \Lambda(\overline{Z}_1)$ such that $\eta_0 \subseteq \lambda_0$ holds and η_0 has a facet η_{01} inside $\lambda_0 \cap \lambda_1$. Since $\eta_0 \not\subseteq \text{cone}(w_j; j \neq i)$ holds, we have $w_i \in \eta_0$. Let $\eta_1 \in \Lambda(\overline{Z}_1)$ be the fulldimensional cone with $\eta_{01} \preceq \eta_1$ and $\eta_1 \subseteq \lambda_1$. Then Lemma 6.6 tells us that the pair η_0, η_1 defines a stellar subdivision $\Sigma_1 \rightarrow \Sigma_0$ of fans at $v_i = P(e_i)$.

Let Z_i be the toric varieties associated to Σ_i , and let $\pi: Z_1 \rightarrow Z_0$ denote the toric contraction morphism associated to the stellar subdivision $\Sigma_1 \rightarrow \Sigma_0$. Let $W_i \subseteq \overline{Z}_i$ denote the sets of semistable points corresponding to the chambers η_i , and let $q_i: W_i \rightarrow Z_i$ be the quotient maps. Then we have $W_1 = \widehat{Z}_1$ and $q_1 = p_1$. Moreover, in Z_0 , we observe

$$q_0(\overline{X}_1 \cap W_0) = \overline{\pi(q_1(\overline{X}_1 \cap \overline{T}_1))} = X_0,$$

where $\overline{T}_1 \subseteq \overline{Z}_1$ denotes the big torus. In particular, we see that X_0 is the good quotient space of an open subset of \overline{X}_1 and thus is normal. As it arises from a full dimensional chamber, it is moreover \mathbb{Q} -factorial. Finally, Theorem 5.12 guarantees that $\pi: Z_1 \rightarrow Z_0$ is a neat controlled ambient modification. \square

Combining this Proposition with the descriptions of the moving cone and the semiample cone given in Proposition 4.1, we obtain the following characterizations of combinatorial minimality.

Corollary 6.8. *A \mathbb{Q} -factorial projective variety with finitely generated Cox ring is combinatorially minimal if and only if its effective cone and its moving cone coincide.*

Corollary 6.9. *A \mathbb{Q} -factorial projective surface with finitely generated Cox ring is combinatorially minimal if and only if its effective cone and its semiample cone coincide.*

Proof of Theorem 6.2. We may assume that X arises from a bunched ring. Then any contractible class $[D]$ on X defines an exceptional weight. By a suitable small birational transformation, we achieve that the semiample cone of X has a common facet with a full dimensional chamber containing this exceptional weight. Now we can use Proposition 6.7 to contract the class $[D]$ and obtain a \mathbb{Q} -factorial projective variety with finitely generated Cox ring. Repeating this procedure, we finally arrive at a combinatorially minimal variety. \square

In order to underline the computational nature of combinatorial contraction, we discuss an explicit example; it provides a non-toric surface that contracts to the projective plane.

Example 6.10. We consider again the \mathbb{Q} -factorial projective surface of Example 4.17. This time, we call it X_1 ; it arises from the bunched ring

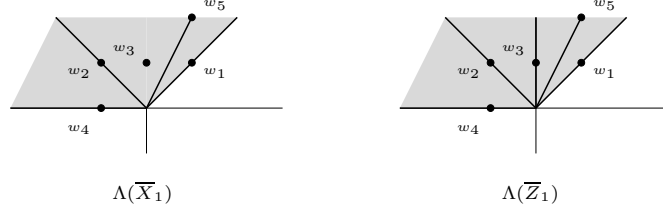
$$R = \mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2 + T_3^2 + T_4 T_5 \rangle,$$

with the system \mathfrak{F} consisting of the classes $f_i \in R$ of $T_i \in \mathbb{K}[T_1, \dots, T_5]$. The K -grading of R is given by $\deg(f_i) := w_i$, with the columns w_i of the matrix

$$Q_1 := \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}$$

and the \mathfrak{F} -bunch Φ consists of the ample cone $\tau = \text{cone}(w_2, w_5)$. The weight $w_4 \in K$ is extremal, and thus there must be a contraction.

To determine the contraction explicitly, we need an associated neat embedding into a toric variety as in Remark 6.3. The two GIT-fans look as follows; the shaded area indicates $Q^0(\gamma)$.



We choose as toric ambient variety of X_1 the toric variety Z_1 defined by the GIT-cone generated by w_2 and w_3 . Consider

$$P_1 := \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Then the rays of the fan Σ_1 of Z_1 are the rays through the columns v_1, \dots, v_5 and the maximal cones of Σ_1 are explicitly given as

$$\begin{aligned} &\text{cone}(v_1, v_2, v_3), & \text{cone}(v_1, v_2, v_5), & \text{cone}(v_1, v_3, v_4), \\ &\text{cone}(v_1, v_4, v_5), & \text{cone}(v_2, v_3, v_5), & \text{cone}(v_3, v_4, v_5). \end{aligned}$$

The toric ambient modification $\Sigma_1 \rightarrow \Sigma_0$ corresponding to the chambers $\text{cone}(w_2, w_4)$ and $\text{cone}(w_3, w_2)$ of the fan $\Lambda(\overline{Z}_1)$ contracts the ray v_4 . Thus, the primitive generators of Σ_0 are the vectors v_1, v_2, v_3, v_5 and its maximal cones are

$$\text{cone}(v_1, v_2, v_3), \quad \text{cone}(v_1, v_2, v_5), \quad \text{cone}(v_2, v_3, v_5), \quad \text{cone}(v_1, v_3, v_5).$$

Note that the last one contains the vector v_4 ; in fact, we have $v_4 = 2v_1 + v_3 + 3v_5$. The index of the stellar subdivision $\Sigma_1 \rightarrow \Sigma_0$ equals one, and the associated toric morphism $\overline{\pi}: \overline{Z}_1 \rightarrow \overline{Z}_0$ of the toric total coordinate spaces is given by

$$(z_1, \dots, z_5) \mapsto (z_1 z_4^2, z_2, z_3 z_4, z_4^3 z_5).$$

According to Theorem 5.12, the total coordinate space of X_0 is given as $\overline{X}_0 = \overline{\pi}(\overline{X}_1)$. One easily checks that the vanishing ideal \overline{X}_0 in $\overline{Z}_0 = \mathbb{K}^4$ is generated by the polynomial

$$T_1 T_2 + T_3^2 + T_4.$$

In particular, we see that the total coordinate space \overline{X}_0 is the affine 3-space; it can explicitly be parametrized by

$$\mathbb{K}^3 \rightarrow \overline{X}_0, \quad (w_1, w_2, w_3) \rightarrow (w_1, w_2, w_3, -w_1 w_2 - w_3^2).$$

Using this, we obtain that the induced action of $H_0 \cong \mathbb{K}^*$ is scalar multiplication. Thus, X_0 is a projective plane. Note that the map $X_1 \rightarrow X_0$ contracts the smooth rational curve $p_1(V(\widehat{X}_0; T_4)) \subseteq X_1$ containing the singular point of X_1 .

7. EXAMPLE: COX RINGS WITH A SINGLE RELATION

Here we apply the results on toric ambient modifications to Cox rings defined by a single relation. We give a simple criterion for neat controlled ambient modifications in this setting, and we explicitly determine the resulting Cox rings; this can be used to construct new factorial hypersurfaces out of given ones. As a concrete example, we perform the minimal resolution of a singular del Pezzo surface via toric ambient resolution.

The setup of this section is the following. Let K_0 be a finitely generated abelian group, and endow the polynomial ring $\mathbb{K}[T_1, \dots, T_r]$ with a K_0 -grading by setting

$$\deg(T_i) := w_i, \quad \text{where } w_1, \dots, w_r \in K_0.$$

Set $\bar{Z}_0 := \mathbb{K}^r$ and $H_0 := \text{Spec}(\mathbb{K}[K_0])$, and suppose that $\hat{Z}_0 \rightarrow Z_0 = \hat{Z}_0 // H_0$ are the data of a toric Cox construction. Then $\hat{Z}_0 \rightarrow Z_0$ is given by a map $\hat{\Sigma}_0 \rightarrow \Sigma_0$ of fans living in lattices $F_0 = \mathbb{Z}^r$ and N_0 . We assume that $\hat{\Sigma}_0$ is a maximal projectable fan; in particular, Σ_0 cannot be enlarged without adding new rays.

Let $f_0 \in \mathbb{K}[T_1, \dots, T_r]$ be homogeneous with respect to the K_0 -grading and suppose that it defines a factorially K_0 -graded residue algebra

$$R := \mathbb{K}[T_1, \dots, T_r] / \langle f_0 \rangle$$

such that the classes $T_i + \langle f_0 \rangle$ are pairwise nonassociated K_0 -prime elements in R . Consider

$$\bar{X}_0 := V(f_0), \quad \hat{X}_0 := \bar{X}_0 \cap \hat{Z}_0, \quad X_0 := \hat{X}_0 // H \subseteq Z_0.$$

Then X_0 is a normal variety having the H_0 -variety \bar{X}_0 as its total coordinate space, and $X_0 \rightarrow Z_0$ is a neat embedding, see Proposition 3.14.

Let v_1, \dots, v_r be the primitive lattice vectors in the rays of Σ_0 . We suppose that for $2 \leq d \leq r$, the cone σ_0 generated by v_1, \dots, v_d belongs to Σ_0 and consider the stellar subdivision $\Sigma_1 \rightarrow \Sigma_0$ at a vector

$$v_\infty = a_1 v_1 + \dots + a_d v_d.$$

Denote the index of this subdivision by m_∞ and the associated toric modification by $\pi: Z_1 \rightarrow Z_0$. Then we obtain the strict transform $X_1 \subseteq Z_1$ mapping onto $X_0 \subseteq Z_0$. Moreover, as in Section 5, we have the commutative diagrams

$$\begin{array}{ccc} & \bar{Z}_1 & \\ \pi_1 \swarrow & & \searrow \pi \\ \bar{Z}_1 & \hat{Z}_1 & \bar{Z}_0 \\ \hat{\pi}_1 \swarrow & & \searrow \hat{\pi} \\ \hat{Z}_1 & & \hat{Z}_0 \\ p_1 \downarrow / H_1 & & \downarrow / H_0 p_0 \\ Z_1 & \xrightarrow{\pi} & Z_0 \end{array} \quad \begin{array}{ccc} & \bar{Y}_1 & \\ \bar{\kappa}_1 \swarrow & & \searrow \bar{\kappa} \\ \bar{X}_1 & \hat{Y}_1 & \bar{X}_0 \\ \hat{\kappa}_1 \swarrow & & \searrow \hat{\kappa} \\ \hat{X}_1 & & \hat{X}_0 \\ p_1 \downarrow / H_1 & & \downarrow / H_0 p_0 \\ X_1 & \xrightarrow{\kappa} & X_0 \end{array}$$

where $H_1 = \text{Spec}(\mathbb{K}[K_1])$ for $K_1 = E_1/M_1$ in analogy to $K_0 = E_0/M_0$; see Section 2 for the notation. In particular, the Cox ring $\mathbb{K}[T_1, \dots, T_r, T_\infty]$ of \bar{Z}_1 comes with a K_1 -grading. Recall moreover, that with respect to the coordinates corresponding to the rays of the fans Σ_i , the map $\bar{\pi}: \bar{Z}_1 \rightarrow \bar{Z}_0$ is given by

$$\bar{\pi}(z_1, \dots, z_r, z_\infty) = (z_\infty^{a_1} z_1, \dots, z_\infty^{a_d} z_d, z_{d+1}, \dots, z_r).$$

We want to give an explicit criterion for $Z_1 \rightarrow Z_0$ to be a neat controlled ambient modification for $X_0 \subset Z_0$, and, in this case, describe the total coordinate space of X_1 explicitly. For this, consider the grading

$$\mathbb{K}[T_1, \dots, T_r] = \bigoplus_{k \geq 0} \mathbb{K}[T_1, \dots, T_r]_k, \quad \text{where } \deg(T_i) := \begin{cases} a_i & i \leq d, \\ 0 & i \geq d+1. \end{cases}$$

Then we may write $f_0 = g_{k_0} + \dots + g_{k_m}$ where $k_0 < \dots < k_m$ and each g_{k_i} is a nontrivial polynomial having degree k_i with respect to this grading.

Definition 7.1. We say that the polynomial $f_0 \in \mathbb{K}[T_1, \dots, T_r]$ is *admissible* if

- (i) the toric orbit $0 \times \mathbb{T}^{r-d}$ intersects $\overline{X}_0 = V(f_0)$,
- (ii) g_{k_0} is a K_1 -prime polynomial in at least two variables.

Note that for the case of a free abelian group K_1 , the second condition just means that g_{k_0} is an irreducible polynomial.

Proposition 7.2. *If, in the above setting, the polynomial f_0 is admissible, then the following holds.*

- (i) *The toric morphism $Z_1 \rightarrow Z_0$ is a neat controlled toric ambient modification for $X_0 \subseteq Z_0$ and $X_1 \subseteq Z_1$.*
- (ii) *The strict transform $X_1 \subseteq Z_1$ is a neatly embedded normal subvariety with total coordinate space \overline{X}_1 and Cox ring*

$$\mathcal{R}(X_1) = \mathbb{K}[T_1, \dots, T_r, T_\infty] / \langle f_1(T_1, \dots, T_r, \sqrt[m]{T_\infty}) \rangle,$$

where in

$$f_1 := \frac{f_0(T_\infty^{a_1} T_1, \dots, T_\infty^{a_d} T_d, T_{d+1}, \dots, T_r)}{T_\infty^{k_0}} \in \mathbb{K}[T_1, \dots, T_r, T_\infty]$$

only powers $T_\infty^{lm_\infty}$ with $l \geq 0$ of T_∞ occur, and the notation $\sqrt[m]{T_\infty}$ means replacing $T_\infty^{lm_\infty}$ with T_∞^l in f_1 .

Lemma 7.3. *Suppose that the polynomial ring $\mathbb{K}[T_1, \dots, T_r, T_\infty]$ is graded by some finitely generated abelian group K_1 . Let $f_1 = gT_\infty + h \in \mathbb{K}[T_1, \dots, T_r, T_\infty]$ be irreducible with $g \in \mathbb{K}[T_1, \dots, T_r, T_\infty]$ and a K_1 -prime $h \in \mathbb{K}[T_1, \dots, T_r]$. Then, for $\overline{X}_1 := V(f_1)$ and $H_1 := \text{Spec}(\mathbb{K}[K_1])$, the intersection $\overline{X}_1 \cap V(T_\infty)$ is H_1 -irreducible, and $\text{Sing}(\overline{X}_1) \cap V(T_\infty)$ is a proper subset of $\overline{X}_1 \cap V(T_\infty)$.*

Proof. We only have to show that $A := V(T_\infty, f_1, \partial f_1 / \partial T_1, \dots, \partial f_1 / \partial T_r)$ is a proper subset of $B := V(f_1, T_\infty)$. A simple calculation gives

$$\text{grad}(f_1) = \left(\frac{\partial g}{\partial T_1} T_\infty + \frac{\partial h}{\partial T_1}, \dots, \frac{\partial g}{\partial T_r} T_\infty + \frac{\partial h}{\partial T_r}, \frac{\partial g}{\partial T_\infty} T_\infty + g \right).$$

Thus, setting $T_\infty = 0$, we see that $A = B$ implies vanishing of $\text{grad}(h)$ along $V(h)$. This contradicts to the assumption that h is K_1 -prime. \square

Proof of Proposition 7.2. We first show that $\overline{X}_1 \subseteq \overline{Z}_1$ is normal. Recall that $\overline{\kappa}: \overline{Y}_1 \rightarrow \overline{X}_0$ is a good quotient for the \mathbb{K}^* -action on \overline{Y}_1 . Outside $V(T_\infty)$, this action is free and \overline{Y}_1 locally splits as $\overline{X}_0 \times \mathbb{K}^*$. In particular, $\overline{Y}_1 \setminus V(T_\infty)$ inherits normality from \overline{X}_0 . Let $\overline{T}_0 \subseteq \overline{X}_0$ and $\overline{T}_1 \subseteq \overline{X}_1$ be the big tori. Then we have

$$\overline{Y}_1 = \overline{\kappa}^{-1}(\overline{X}_0 \cap \overline{T}_0) = \overline{V}(\overline{T}_1; \overline{\kappa}^* f_0) = V(\overline{Z}_1; f_1).$$

Combining this with Lemma 7.3, we see that \overline{Y}_1 is regular in codimension one. Thus, Serre's criterion shows that \overline{Y}_1 is normal. Moreover, the group C_{m_∞} of roots of unity acts on \overline{Z}_1 via multiplication on the last coordinate, and \overline{Y}_1 is invariant under this action. Thus, f_1 is C_{m_∞} -homogeneous. Writing

$$f_1 = \sum_{i=0}^m \frac{g_{k_i}(T_\infty^{a_1} T_1, \dots, T_\infty^{a_d} T_d, T_{d+1}, \dots, T_r)}{T_\infty^{k_0}},$$

we see that the g_{k_0} -term is invariant under C_{m_∞} , and hence the others must be as well; note that in different g_{k_i} , the variable T_∞ occurs in different powers. Thus, only powers $T_\infty^{lm_\infty}$ of T_∞ occur in f_1 . Now, \overline{X}_1 is the quotient of \overline{Y}_1 by the action of C_{m_∞} . Consequently, \overline{X}_1 is normal, and we obtain

$$\overline{X}_1 = V(\overline{Z}_1; f_1(T_1, \dots, T_r, \sqrt[m]{T_\infty})).$$

Now the assertions drop out. First, as a quotient of the normal variety $\widehat{X}_1 \subseteq \overline{X}_1$, the variety X_1 is again normal. Since $\overline{X}_1 \cap V(T_\infty)$ is H_1 -irreducible, the exceptional

locus of $X_1 \rightarrow X_0$, as the image of $\widehat{X}_1 \cap V(T_\infty)$ under the quotient $\widehat{X}_1 \rightarrow X_1$ by H_1 , is irreducible. Moreover, note that

$$\Gamma(\overline{X}_1, \mathcal{O}) / \langle T_{\infty| \overline{X}_1} \rangle \cong \mathbb{K}[T_1, \dots, T_r, T_\infty] / \langle f_1, T_\infty \rangle \cong \mathbb{K}[T_1, \dots, T_r] / \langle g_{k_0} \rangle.$$

We can conclude that the restriction of T_∞ to \overline{X}_1 provides a local equation for $\overline{X}_1 \cap V(T_\infty)$. Thus, we verified all conditions for $Z_1 \rightarrow Z_0$ to be a neat ambient modification. Consequently, Corollary 5.10 shows that \overline{X}_1 is the total coordinate space of X_1 . \square

Proposition 7.2 may as well be used to produce multigraded factorial rings, as the following statement shows.

Corollary 7.4. *The ring $R := \mathbb{K}[T_1, \dots, T_r, T_\infty] / \langle f_1(T_1, \dots, T_r, \sqrt[m]{T_\infty}) \rangle$ in Proposition 7.2 is factorially K_1 -graded. Moreover, if K_0 is torsion free, then R is even factorial.*

As an example, we consider a \mathbb{Q} -factorial projective surface X_0 with a single singularity, and resolve this singularity by means of toric ambient resolution, showing thereby that X_0 is a singular del Pezzo surface.

Example 7.5. We consider once more the \mathbb{Q} -factorial projective surface arising from the bunched ring (R, \mathfrak{F}, Φ) of 4.17. This time, we call it X_0 . Recall that we have

$$R = \mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2 + T_3^2 + T_4 T_5 \rangle,$$

the system \mathfrak{F} consists of the classes $\overline{T}_i \in R$ of $T_i \in \mathbb{K}[T_1, \dots, T_5]$. Moreover, the K_0 -grading of R is given by $\deg(\overline{T}_i) := w_i$, with the columns w_i of the matrix

$$Q_0 := \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}$$

and the \mathfrak{F} -bunch Φ consists of the ample cone $\tau = \text{cone}(w_2, w_5)$. As in 6.10, we take the toric variety Z_0 defined by the GIT-cone generated by w_2 and w_3 as the toric ambient variety of X_0 . Then the columns v_1, \dots, v_5 of

$$P_0 := \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

generate the rays of Σ_0 , and the maximal cones of Σ_0 are explicitly given as

$$\begin{aligned} &\text{cone}(v_1, v_2, v_3), & \text{cone}(v_1, v_2, v_5), & \text{cone}(v_1, v_3, v_4), \\ &\text{cone}(v_1, v_4, v_5), & \text{cone}(v_2, v_3, v_5), & \text{cone}(v_3, v_4, v_5). \end{aligned}$$

Note that X_0 inherits its singularity from Z_0 ; it is the toric singularity corresponding to the cone generated by v_1, v_3 and v_4 . In view of Corollary 4.13, we may obtain a resolution of the singularity of X by resolving the ambient singularity.

Resolving the toric ambient singularity corresponding to $\text{cone}(v_1, v_3, v_4)$ means to successively subdivide this cone along the interior members of the Hilbert basis. These are $v_6 := (0, -1, -1)$ and $v_7 := (1, -1, -1)$; note that the order does not influence the result on the resolution obtained for X_0 .

We start with subdividing in $v_6 = (0, -1, -1)$. Up to renumbering coordinates, we are in the setting of Proposition 7.2. Note that the index of this subdivision is $m_6 = 3$. The pullback relation on \overline{Z}_1 is $T_1 T_2 + T_3^2 T_6^3 + T_4 T_5$. Dividing by the C_3 -action, we obtain

$$f_1 := T_1 T_2 + T_3^2 T_6 + T_4 T_5 \in \mathbb{K}[T_1, \dots, T_6].$$

for the relation in the Cox ring of the modified surface $X_1 \subseteq Z_1$. This still has a singular point, namely the toric fixed point corresponding to the cone generated by v_1, v_4 and v_6 .

In the next step, we have to subdivide by v_7 . Note that now we have index $m_7 = 2$. The pullback relation on \overline{Z}_2 is $T_1T_2 + T_3^2T_6^3 + T_4T_5$. It doesn't depend on the new variable T_7 , hence, dividing by the C_2 -action, we get

$$f_2 := T_1T_2 + T_3^2T_6 + T_4T_5 \in \mathbb{K}[T_1, \dots, T_7]$$

for the defining relation of the Cox ring of the modified surface $X_2 \subseteq Z_2$. In order to see that $X_2 \rightarrow X_0$ is a resolution, we need to know that \widehat{X}_2 is smooth. But this is due to the fact that the singular locus of \overline{X}_2 is described by $T_6 = 0$ and w_6 is exceptional. Note that the grading of the Cox ring of X_2 is given by assigning to T_i the i -th column of the matrix

$$Q_2 := \begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 \end{bmatrix}$$

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